Bonus Problems

Problems from BMT 2016.

1. (A4) A geometric progression starting at $a_0 = 3$ has an even number of terms. Suppose the difference between the odd-indexed terms and the even-indexed terms is 39321 and the sum of the first and last terms is 49155. Find the common ratio of this progression.

2. (D6) Bob plays a game on the whiteboard. Initially, the numbers $1, 2, \ldots, n$ are written. On each turn, Bob erases the numbers x, y on the board and writes down $2x + y$, and he repeats this process until only one number remains. In terms of n, what is the maximum possible remaining value?

3. (G7) Let ABC be a right triangle with $AB = BC = 2$. Construct point D such that $\angle DAC = 30°$ and $\angle DCA = 60°$, and $\angle BCD > 90°$. Compute the area of triangle BCD.

4. (G9) Given right triangle ABC with right angle at C, construct three external squares $ABDE, BCFG$, and $ACHI$. If $DG = 19$ and $EI = 22$, compute the length of FH.

5. (A10) Evaluate
$$\sum_{k=0}^{\infty} \left(\frac{-1}{8}\right)^k \binom{2k}{k}.$$

6. (D10) An $m \times n$ rectangle is tiled with 1×2 dominoes such that whenever the rectangle is partitioned into two smaller rectangles, there exists a domino that is part of the interior of both rectangles. Given $mn > 2$, what is the minimum possible value of mn?

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1. (China 2014) Let ABC be a triangle with $AB > AC$. Let D be the foot of the angle bisector of A. Points F and E are on AC, AB, respectively, such that $BCFE$ is cyclic. Prove that the circumcenter of DEF is the incenter of ABC if and only if $BE + CF = BC$.

2. (China 2013) Find all nonempty sets S of integers such that $3m - 2n \in S$ for any pair of not necessarily distinct elements $m, n \in S$.