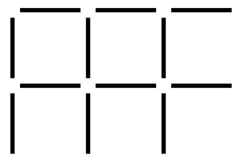
Short Math Contests	Misha Lavrov

Problems from the 2012 St. Petersburg Math Olympiad

Western PA ARML Practice — *October 23, 2016*

1. Eeyore has 2012 sticks of length 1 cm, which he uses to make a rectangular grid in which each cell is a 1 cm by 1 cm square. If P is the perimeter of the grid and S its area, find $P + 4S$. (Below is an example 2×3 grid made up of 17 sticks.)



2. A sequence of k consecutive integers $a+1, a+2, \ldots, a+k$ is written on a chalkboard. If exactly 52% of the integers are even, find k.

3. The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\sin 42°$ and $\sin 48°$. Prove that $b^2 = a^2 + 2ac$.

4. Is it possible to arrange the integers $1, 2, \ldots, 100$ around a circle such that for every pair of adjacent integers x and y, at least one of the quantities $x - y, y - x, \frac{x}{y}, \frac{y}{x}$ is equal to 2?

5. In $\triangle ABC$, BL is an angle bisector and $\angle C = 3\angle A$. Point M on AB and point N on AC are chosen so that $\angle AML = \angle ANM = 90°$. Prove that $BM + 2MN > BL + LM$.

6. Prove that for distinct real numbers a, b, c, the system of equations

$$\begin{cases} x^3 - ax^2 + b^3 = 0 \\ x^3 - bx^2 + c^3 = 0 \\ x^3 - cx^2 + a^3 = 0 \end{cases}$$

has no real solutions.

7. Circles ω_1 and ω_2 are externally tangent at P. Line ℓ_1 passes through the center of ω_1 and is tangent to ω_2; similarly, line ℓ_2 passes through the center of ω_2 and is tangent to ω_1. If ℓ_1 and ℓ_2 intersect at X, prove that XP bisects one of the angles formed at X between ℓ_1 and ℓ_2.

8. Peter chose a natural number $n > 1$ and wrote the numbers

$$1 + n, 1 + n^2, 1 + n^3, \ldots, 1 + n^{15}$$

on a chalkboard. Then he erased some of the numbers so that among the remaining numbers, any two are relatively prime. At most how many numbers could Peter have left on the board?

9. Let a and b be two distinct positive integers. The equations $y = \sin ax$ and $y = \sin bx$ are graphed in the same coordinate plane, and all of their intersection points are marked. Prove that there is a third positive integer c, distinct from a and b, such that the graph of $y = \sin cx$ passes through all the marked points.