Exam Version 2

Problems compiled from Stanford Math Tournament 2014.

1. (AT 5) Compute the number of ways there are to select three distinct lattice points in three-dimensional space such that the three points are collinear and no point has a coordinate with absolute value exceeding 1.

2. (A 6) Find the minimum value of
$$\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{x-z}$$
for reals $x > y > z$ given that $(x-y)(y-z)(x-z) = 17$.

3. (G 7) Let ABC be a triangle th $AB = 13, BC = 14$, and $AC = 15$. Let D and E be the feet of the altitudes from A and B, respectively. Find the circumference of the circumcircle of $\triangle CDE$.

4. (AT 8) Let $a_0, a_1, \ldots,$ be a sequence of positive integers where $a_n = n!$ for $n \leq 3$, and, for $n \geq 4$, a_n is the smallest positive integer such that
$$\frac{a_n}{a_i a_{n-i}}$$
is an integer for all $0 \leq i \leq n$. Find a_{2014}.

5. (A 8) P and Q are polynomials such that
$$P(P(x)) = P(x)^{16} + x^{48} + Q(x).$$
Compute the smallest possible degree of Q.

6. (G 8) Let O be a circle of radius 1. A and B are fixed points on the circle such that $AB = \sqrt{2}$. Let C be any point on the circle, and let M and N be the midpoints of AC and BC, respectively. As C travels around the circle O, find the area of the locus of points on MN.

7. (AT 9) Compute the smallest positive integer n such that the leftmost digit of 2^n (in base 10) is 9.

8. (A 9) Let b_n be a sequence defined by the formula
$$b_n = \sqrt[3]{-1 + a_1\sqrt[3]{-1 + a_2\sqrt[3]{-1 + \ldots a_{n-1}\sqrt[3]{-1 + a_n}}}},$$
where a_n is given by $a_n = n^2 + 3n + 3$. Find the smallest real number L such that $b_n < L$ for all n.

9. (G 10) Let ABC be a triangle with $AB = 12$, $BC = 5$, $AC = 13$. Let D and E be the feet of the internal and external angle bisectors from B, respectively. (The external angle bisector from B bisects the angle between BC and the extension of AB.) Let ω be the circumcircle of $\triangle BDE$; extend AB so that it intersects ω again at F. Extend FC to meet ω again at X, and extend AX to meet ω again at G. Find FG.