Team Contest

Problems compiled from Stanford Math Tournament 2012 and 2013.

1. (2012 #1) How many functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ take on exactly 3 distinct values?

2. (2013 #1) Let $f_1(n)$ be the number of divisors that n has, and define $f_k(n) = f(1)(f_{k-1}(n))$ for $k > 1$. Compute the smallest integer k such that $f_k(2013^{2013}) = 2$.

3. (2013 #2) In unit square $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. Let M be the midpoint of $\overline{CD}$ with $\overline{AM}$ intersecting $\overline{BD}$ at F and $\overline{BM}$ intersecting $\overline{AC}$ at G. Find the area of quadrilateral $MFEG$.

4. (2013 #3) Nine people are practicing the triangle dance, which is a dance that requires a group of three people. During each round of practice, the nine people split off into three groups of three people each, and each group ractices independently. Two rounds of practice are different if there exists some person who does not dance with the same pair in both rounds. How many different rounds of practice can take place?

5. (2012 #5) Regular hexagon $A_1A_2A_3A_4A_5A_6$ has side length 1. For $1 \leq i \leq 6$, choose B_i to be a point on the segment A_iA_{i+1} uniformly at random, where $A_7 = A_1$. What is the expected value of the area of hexagon $B_1B_2B_3B_4B_5B_6$?

6. (2013 #6) How many distinct sets of five distinct positive integers A satisfy the property that for any positive integer $x \leq 29$, there exists a subset $S \subseteq A$ such that the sum of elements in S is x?

7. (2012 #6) Evaluate
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm(n+m+1)}.$$

8. (2013 #11) What is the smallest positive integer with exactly 768 divisors? You may leave your answer in its prime factorization.

9. (2012 #12) A triangle has sides of length $\sqrt{2}$, $3 + \sqrt{3}$, and $2\sqrt{2} + \sqrt{6}$. Compute the area of the smallest regular polygon that has three vertices coinciding with the vertices of the given triangle.

10. (2013 #14) You have a 2 meter long string. You choose a point along the string uniformly at random and make a cut, then discard the shorter section. If you still have at least 0.5 meters of string remaining, you repeat this process; if you have less than 0.5 meters of string, you stop. What is the expected number of cuts you will make before stopping?