Select Problems from SDML 2012-2013

On the actual SDML, there are eight problems to do in 48 minutes. As such, if you're looking for a challenge, attempt problems 3 through 10; otherwise, work on problems 1 through 8. You have 50 minutes. Good luck!

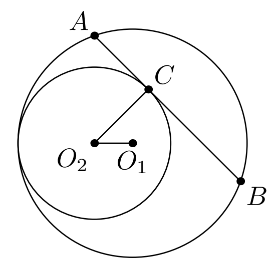
1. If five boys and three girls are randomly divided into two four-person teams, what is the probability that all three girls will end up on the same team?

2. Let $b = \log_5 3$. What is $\log_b(\log_3 5)$?

3. Palmer correctly computes the product of the first $1,001$ prime numbers. Which of the following is NOT a factor of Palmer's product?

   (A) $2,002$ **(B)** $3,003$ **(C)** $5,005$ **(D)** $6,006$ **(E)** $7,007$

4. A scientist begins an experiment with a cell culture that starts with some integer number of identical cells. After the first second, one of the cells dies, and every two seconds from there another cell will die (so one cell dies every odd-numbered second from the starting time). Furthermore, after exactly 60 seconds, all of the living cells simultaneously split into two identical copies of itself, and this continues to happen every 60 seconds thereafter. After performing the experiment for awhile, the scientist realizes the population of the culture will be unbounded and quickly shuts down the experiment before the cells take over the world. What is the smallest number of cells that the experiment could have started with?



5. Naoki's favorite positive integer n is a two-digit number with distinct digits. It also has the property that when it is divided by 10, 12, and 14, the remainder has a units digit of one. What is the value of n?

6. Circle ω_1 with center O_1 has radius 3, and circle ω_2 with center O_2 has radius 2 and is internally tangent to ω_1. The segment AB is a chord of ω_1 that is tangent to ω_2 at C with $\angle O_1O_2C = 45°$. Find the length of AB.

7. The game tic-tac is played on a 3 by 3 square grid between players X and O. They take turns, and on their turn a player writes their symbol onto one empty space of the grid. A player wins if they fill a row or column with three copies of their symbol; a player filling a main diagonal does not end the game in a win for that player. If the grid is filled without determining the winner, the game is a draw. Assuming player X goes first and the players draw the game, how many possibilities are there for the final state of the grid?

8. A finite arithmetic progression of positive integers $a_1, a_2, \ldots, a_n$ satisfies the condition that for all $1 \le i < j \le n$, the number of positive divisors of $\gcd(a_i, a_j)$ is equal to $j - i$. Find the maximum possible value of n.

9. Let a, b, c, d be real numbers. Suppose that

$$\frac{a}{b+c} + \frac{b}{a+d} = \frac{3}{5}, \qquad \frac{b}{c+d} + \frac{c}{a+b} = 1, \qquad \frac{c}{a+d} + \frac{d}{b+c} = \frac{7}{5}.$$

   Find the value of

$$\frac{d}{a+b} + \frac{a}{c+d}.$$

10. Let ℓ be a line in the plane. Two circles with respective radii 2 and 4 are tangent to ℓ on the same side so that their points of tangency are distance 9 apart. The two common internal tangents to both circles are drawn. What is the area of the triangle formed by the line ℓ and the two internal tangents?[1]

[1] When I originally solved this problem, I misread the question and thought that the distance between the centers of the circles was 9. Thus, my original solution tried to optimize calculations with this in mind. While it does contain a very heavy dose of geometry, I do think it is instructive, so I may present it if I have time.