Sequences and Series	Misha Lavrov

Solving Recurrences

Western PA ARML Practice — *December 11, 2016*

0 Useful Facts

Solving linear recurrences. To the recurrence $x_n = Ax_{n-1} + Bx_{n-2}$ is associated the characteristic equation $x^2 = Ax + B$. If this equation has two[1] distinct[2] roots r_1, r_2, then $x_n = r_1^n$ and $x_n = r_2^n$ are solutions to the recurrence, and all other solutions have the form $x_n = C_1 r_1^n + C_2 r_2^n$.

Fixed points. *If* the recurrence $x_n = f(x_{n-1})$ (where f is an arbitrary function) eventually converges, approaching a fixed limit x, then that limit must satisfy $x = f(x)$.

Monotonicity. If a sequence (x_n) satisfies $x_n < x_{n+1}$ and $x_n < M$ for all n (if it keeps growing, but never passes some fixed upper bound) then it converges.

1 Exercises

1. Solve the recurrence $a_n = 3a_{n-1} + 1$ with $a_0 = 0$.
2. Solve the recurrence $b_n = 3b_{n-1} + n$ with $b_0 = 0$.
3. Solve the recurrence $c_n = 3c_{n-1} + 2^n$ with $c_0 = 0$.
4. Solve the recurrence $d_n = d_{n-1} + d_{n-2} + 1$ with $d_0 = 1$ and $d_1 = 2$.
5. Solve the recurrence $e_n = e_{n-1} + 2e_{n-2}$ with $e_0 = 0$ and $e_1 = 1$.

2 Problems

6. Compute
$$2 + \cfrac{1}{2 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\cdots}}}}}}.$$

[1] We can use the same method to solve equations that go back more than two terms, but then the characteristic equation is cubic or worse.

[2] If the equation instead has a double root r, then $x_n = r^n$ and $x_n = n \cdot r^n$ are solutions, and all other solutions have the form $x_n = (C_1 + C_2 n) \cdot r^n$.

7. Compute

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}}.$$

8. Compute

$$(-2+(-2+(-2+(-2+(-2+(-2+\cdots)^2)^2)^2)^2)^2)^2.$$

9. The Fibonacci numbers are defined by the recurrence $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0$ and $F_1 = 1$. Find a recurrence for the squared Fibonacci numbers F_n^2.

10. (a) Compute $\left(\frac{1+\sqrt{13}}{2}\right)^{10} + \left(\frac{1-\sqrt{13}}{2}\right)^{10}$.

    (b) Compute the remainder $\left(\frac{1+\sqrt{13}}{2}\right)^{1000} + \left(\frac{1-\sqrt{13}}{2}\right)^{1000}$ mod 18.

11. (Concrete Math) Solve the recurrence $2T_n = nT_{n-1} + 3 \cdot n!$, with $T_0 = 5$.

12. (PUMaC 2007) Two sequences x_n and y_n are defined by $x_0 = y_0 = 7$ and

$$\begin{cases} x_n = 4x_{n-1} + 3y_{n-1}, \\ y_n = 3y_{n-1} + 2x_{n-1}. \end{cases}$$

    Find the limiting value of $\frac{x_n}{y_n}$ as $n \to \infty$.

13. The Lucas–Lehmer test for whether a Mersenne number $2^p - 1$ is prime is to compute the remainder s_{p-2} mod $2^p - 1$, where s_i is a sequence recursively defined by $s_i = s_{i-1}^2 - 2$, with $s_0 = 4$.

    (Specifically, $2^p - 1$ is prime if and only if p is prime and $s_{p-2} \equiv 0 \pmod{2^p - 1}$.)

    Find a closed-form solution for s_n.

14. (Putnam 1985, modified) Solve the recurrence $a_{j+1} = a_j^2 + 2a_j$ with $a_0 = 1$.

15. (VTRMC 1990) The number of individuals in a certain population (in arbitrary real units) obeys, at discrete time intervals, the equation $y_{n+1} = y_n(2 - y_n)$ for $n = 0, 1, 2, \ldots$, where y_0 is the initial population.

    (a) Find all "steady-state" solutions y^* such that, if $y_0 = y^*$, then $y_n = y^*$ for $n = 1, 2, \ldots$.

    (b) Prove that if $0 < y_0 < 1$, then the sequence (y_n) converges monotonically to one of the steady-state solutions found in (a).

16. (Putnam 2016) Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$, $x_{n+1} = \ln(e^{x_n} - x_n)$. Show that the infinite series $x_0 + x_1 + x_2 + \cdots$ converges and find its sum.