Summing Series: Solutions

2 Switching the order of summation

1. *Prove useful identity (9).*

$$\sum_{k=1}^{\infty} kx^k = \sum_{k=1}^{\infty}\sum_{j=1}^{k} x^k = \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} x^k = \sum_{j=1}^{\infty} \frac{x^j}{1-x} = \frac{1}{1-x}\sum_{j=1}^{\infty} x^j = \frac{1}{1-x}\cdot\frac{x}{1-x}.$$

2. *Riemann's zeta function $\zeta(k)$ is defined to be the infinite sum*

$$\zeta(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots = \sum_{j=1}^{\infty} \frac{1}{j^k}.$$

Find $\sum_{k=2}^{\infty}(\zeta(k)-1)$.

$$\sum_{k=2}^{\infty}(\zeta(k)-1) = \sum_{k=2}^{\infty}\sum_{j=2}^{\infty}\frac{1}{j^k} = \sum_{j=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{j^k} = \sum_{j=2}^{\infty}\frac{1/j^2}{1-1/j} = \sum_{j=2}^{\infty}\left(\frac{1}{j-1}-\frac{1}{j}\right) = 1.$$

3. *We define the n^{th}* harmonic number *H_n to be the value of the sum $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, which has no closed form.*

Express $\sum_{k=1}^{n} H_k$ *in terms of H_n.*

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n}\sum_{j=1}^{k}\frac{1}{j} = \sum_{j=1}^{n}\sum_{k=j}^{n}\frac{1}{j} = \sum_{j=1}^{n}\frac{n-j+1}{j} = (n+1)\sum_{j=1}^{n}\frac{1}{j} - \sum_{j=1}^{n} 1 = (n+1)H_n - n.$$

4. *(Concrete Mathematics) Find (again, in terms of H_n)* $\sum_{k=1}^{n}\frac{H_k}{(k+1)(k+2)}$.

$$\sum_{k=1}^{n}\frac{H_k}{(k+1)(k+2)} = \sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{1}{j}}{(k+1)(k+2)} = \sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n}\frac{1}{(k+1)(k+2)}.$$

Applying problem #1 from the next section, we get

$$\sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n}\left(\frac{1}{k+1}-\frac{1}{k+2}\right) = \sum_{j=1}^{n}\frac{1}{j}\left(\frac{1}{j+1}-\frac{1}{n+2}\right) = \sum_{j=1}^{n}\frac{1}{j(j+1)} - \frac{1}{n+2}\sum_{j=1}^{n}\frac{1}{j}.$$

Applying problem #1 from the next section yet again, the first sum in this result becomes $1-\frac{1}{n+1}$, so the whole thing simplifies to $1-\frac{1}{n+1}-\frac{H_n}{n+2}$.

5. *Prove useful identity (6) by writing k^2 as $\sum_{j=1}^{k} k$. (Hint: some things will go wrong, but you can still save the day.)*

Let $\square_n = \sum_{k=1}^{n} k^2$. We have

$$\square_n = \sum_{k=1}^{n}\sum_{j=1}^{k} k = \sum_{j=1}^{n}\sum_{k=j}^{n} k = \sum_{j=1}^{n} \frac{(n+j)(n-j+1)}{2} = \sum_{j=1}^{n} \frac{n^2+n}{2} + \sum_{j=1}^{n} \frac{j}{2} - \sum_{j=1}^{n} \frac{j^2}{2}.$$

Unfortunately, we don't know how to evaluate the very last sum here yet, but we *do* know how to evaluate the other two, and so we get

$$\square_n = \frac{n^3+n^2}{2} + \frac{n^2+n}{4} - \frac{\square_n}{2}.$$

Solving for $\square_n$, we get the formula $\square_n = \frac{n(n+1)(2n+1)}{6}$.

6. *Find $\sum_{k=1}^{n} k \cdot F_k$, where F_n is the n^{th} Fibonacci number: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. (Hint: solve problem #2 in the next section first.)*

$$\sum_{k=1}^{n} k \cdot F_k = \sum_{k=1}^{n}\sum_{j=1}^{k} F_k = \sum_{j=1}^{n}\sum_{k=j}^{n} F_k = \sum_{j=1}^{n}(F_{n+2} - F_{j+1}) = nF_{n+2} - (F_{n+3} - F_3).$$

7. *(ARML 1978) Find the sum of the infinite series $\sum_{k=1}^{\infty} \frac{k^2}{3^k}$.*

$$\sum_{k=1}^{\infty} \frac{k^2}{3^k} = \sum_{k=1}^{\infty}\sum_{j=1}^{k} \frac{k}{3^k} = \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} \frac{k}{3^k}.$$

Let $\ell = k-j+1$, so that we can relabel the inside sum as

$$\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{\ell+j-1}{3^{\ell+j-1}} = \sum_{j=1}^{\infty} \left(\frac{1}{3^{j-1}} \sum_{\ell=1}^{\infty} \frac{\ell}{3^\ell} + \frac{j-1}{3^{j-1}} \sum_{\ell=1}^{\infty} \frac{1}{3^\ell} \right) = \sum_{j=1}^{\infty} \left(\frac{1}{3^{j-1}} \cdot \frac{3}{4} + \frac{j-1}{3^{j-1}} \cdot \frac{1}{2} \right).$$

This we can rewrite as

$$\frac{3}{4} \sum_{j=1}^{\infty} \frac{1}{3^{j-1}} + \frac{1}{2} \sum_{j=1}^{\infty} \frac{j-1}{3^{j-1}} = \frac{3}{4} \cdot \frac{3}{2} + \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{2}.$$

8. *(Putnam 2003) Show that for each positive integer n,*

$$n! = \prod_{j=1}^{n} \operatorname{lcm}\{1, 2, \ldots, \lfloor n/j \rfloor\}.$$

We show that both sides are equal by showing that for any prime p, p divides both sides an equal number of times.

Recall that the number of times p divides $n!$ is $\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$. We can rewrite this as

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = \sum_{k=1}^{\infty} \sum_{j : j < n/p^k} 1,$$

and then reverse the summation to get

$$\sum_{j=1}^{\infty} \sum_{k : p^k < n/j} 1 = \sum_{j=1}^{\infty} \max\{k : p^k < n/j\}.$$

But $\max\{k : p^k < n/j\}$ is precisely the number of times p divides $\operatorname{lcm}\{1, 2, \ldots, \lfloor n/j \rfloor\}$, so the sum we've ended up with is the number of times p divides the product on the left-hand side of the original equation. Therefore we're done.

3 The method of differences

1. *Find the sum* $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \frac{1}{4\cdot5} + \cdots + \frac{1}{98\cdot99} + \frac{1}{99\cdot100}$.

   We can write $\frac{1}{k(k+1)}$ as $\frac{1}{k} - \frac{1}{k+1}$, which means that this sum simplifies to

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right) = 1 - \frac{1}{100} = \frac{99}{100}.$$

2. *Find* $\sum_{k=1}^{n} F_k$*, where* F_n *is the* n^{th} *Fibonacci number:* $F_1 = F_2 = 1$ *and* $F_n = F_{n-1} + F_{n-2}$.

   We can write F_k as $F_{k+2} - F_{k+1}$, so this sum simplifies to

$$(F_3 - F_2) + (F_4 - F_3) + \cdots + (F_{n+1} - F_n) + (F_{n+2} - F_{n+1}) = F_{n+2} - F_2 = F_{n+2} - 1.$$

3. *(Wikipedia) A well-known (but hard) result is that* $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. *Find an approximation for this sum by using the upper bound*

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{\infty} \frac{1}{k^2 - 1/4}$$

and evaluating the sum on the right-hand side. (Bonus: what approximation for π do you get in this way?)

We have $\frac{1}{k^2-1/4} = \frac{1}{(k-1/2)(k+1/2)} = \frac{1}{k-1/2} - \frac{1}{k+1/2}$. So this sum also telescopes to

$$1 + \left(\frac{1}{3/2} - \frac{1}{5/2}\right) + \left(\frac{1}{5/2} - \frac{1}{7/2}\right) + \left(\frac{1}{7/2} - \frac{1}{9/2}\right) + \cdots = 1 + \frac{1}{3/2} = \frac{5}{3}.$$

This is actually very close to the truth: $\frac{\pi^2}{6} \approx 1.645$ and $\frac{5}{3} \approx 1.667$. Relatedly, $\sqrt{10}$ is a pretty good approximation for π.

4. *(ARML 1991) Let* $\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{5^2}\right)\cdots\left(1 - \frac{1}{1991^2}\right) = \frac{x}{1991}$. *Compute the integer x.*

   This is not a telescoping sum but a telescoping product. We can write $1 - \frac{1}{k^2}$ as $\frac{k^2-1}{k^2} = \frac{k-1}{k} \cdot \frac{k+1}{k}$, so the product simplifies to

   $$\left(\frac{2}{3} \cdot \frac{4}{3}\right) \cdot \left(\frac{3}{4} \cdot \frac{5}{4}\right) \cdots \left(\frac{1989}{1990} \cdot \frac{1991}{1990}\right) \cdot \left(\frac{1990}{1991} \cdot \frac{1992}{1991}\right) = \frac{2}{3} \cdot \frac{1992}{1991} = \frac{1328}{1991},$$

   and $x = 1328$.

5. (a) *Prove useful identity (8).*

   Pascal's identity states that $\binom{k}{r} + \binom{k}{r+1} = \binom{k+1}{r+1}$, or $\binom{k}{r} = \binom{k+1}{r+1} - \binom{k}{r+1}$. So we have

   $$\sum_{k=1}^{n} \binom{k}{r} = \sum_{k=1}^{n} \left(\binom{k+1}{r+1} - \binom{k}{r+1}\right) = \binom{n+1}{r+1} - \binom{1}{r+1} = \binom{n+1}{r+1}.$$

   (b) *We have $k^2 = 2\binom{k}{2} + \binom{k}{1}$. Use this, and useful identity (8), to derive useful identity (6).*

   We have
   $$\sum_{k=1}^{n} k^2 = 2\sum_{k=1}^{n} \binom{k}{2} + \sum_{k=1}^{n} \binom{k}{1} = 2\binom{n+1}{3} + \binom{n+1}{2},$$
   which simplifies to $\frac{n(n+1)(2n+1)}{6}$.

   (c) *Find a similar expression for k^3, and use it with useful identity (8) to derive useful identity (7). (Note: this method applies more generally to find the sum of* any *polynomial expression in k.*

   The expression is $k^3 = 6\binom{k}{3} + 6\binom{k}{2} + \binom{k}{1}$, which can be found by repeatedly choosing the right binomial coefficient to subtract that will reduce the degree of the polynomial by 1. From here,

   $$\sum_{k=1}^{n} k^3 = 6\sum_{k=1}^{n} \binom{k}{3} + 6\sum_{k=1}^{n} \binom{k}{2} + \sum_{k=1}^{n} \binom{k}{1} = 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2}.$$

6. (a) *Write the differences* $\sin(n+1) - \sin n$ *and* $\cos(n+1) - \cos n$ *in terms of* $\sin n$, $\cos n$, *and constants.*

We have:

$$\begin{cases} \sin(n+1) - \sin n & = \sin n \cos 1 + \cos n \sin 1 - \sin n \\ & = (-1 + \cos 1) \sin n + \sin 1 \cos n. \\ \cos(n+1) - \cos n & = \cos n \cos 1 - \sin n \sin 1 - \cos n \\ & = (-\sin 1) \sin n + (-1 + \cos 1) \cos n. \end{cases}$$

(b) *Find a function $f(n)$ such that $f(n+1) - f(n) = \sin n$.*

Begin by taking $f(n) = \frac{\sin n}{\sin 1} + \frac{\cos n}{1-\cos 1}$. By the above identities, we'll get

$$f(n+1) - f(n) = \left(\frac{-1+\cos 1}{\sin 1} \sin n + \cos n \right) + \left(\frac{-\sin 1}{1-\cos 1} \sin n - \cos n \right)$$

which simplifies to

$$f(n+1) - f(n) = \left(\frac{-1+\cos 1}{\sin 1} + \frac{-\sin 1}{1-\cos 1} \right) \sin n = -\frac{2}{\sin 1} \sin n.$$

So we can adjust our $f(n)$ by multiplying it by $-\frac{\sin 1}{2}$, getting the new function

$$f(n) = -\frac{1}{2} \sin n - \frac{\sin 1}{2 - 2\cos 1} \cos n.$$

In fact, $f(n)$ further simplifies to $-\frac{\cos(n-\frac{1}{2})}{2 \sin 1/2}$, though we don't need that.

(c) *Find a formula for $\sum_{k=1}^{n} \sin k$.*

Since $\sin k = f(k+1) - f(k)$, this sum telescopes to $f(n+1) - f(1)$, which simplifies to $\frac{\sin \frac{n}{2} \sin \frac{n+1}{2}}{\sin \frac{1}{2}}$.

7. *(VTRMC 2014) Find $\sum_{k=2}^{\infty} \frac{k^2 - 2k - 4}{k^4 + 4k^2 + 16}$.*

Note that $k^4 + 4k^2 + 16 = (k^4 + 8k^2 + 16) - 4k^2 = (k^2+4)^2 - (2k)^2 = (k^2+2k+4) \cdot (k^2-2k+4)$. Furthermore, the summand can be split into the partial fractions

$$\frac{k^2 - 2k - 4}{k^4 + 4k^2 + 16} = \frac{k-2}{2(k^2 - 2k + 4)} - \frac{k}{2(k^2 + 2k + 4)} = Q(k) - Q(k+2),$$

where $Q(x) = \frac{x-2}{2(x^2-2x+4)}$. So the sum telescopes as

$$(Q(2) - Q(4)) + (Q(3) - Q(5)) + (Q(4) - Q(6)) + \cdots = Q(2) + Q(3),$$

since as $x \to \infty$, $Q(x) \to 0$. To find the value of the sum, all we have to do is evaluate $Q(2) + Q(3) = 0 + \frac{1}{2(9-6+4)} = \frac{1}{14}$.

8. *(USAMO 1991) For any set S, let $\sigma(S)$ and $\pi(S)$ denote the sum and product, respectively, of the elements of S, with $\sigma(\emptyset) = 0$ and $\pi(\emptyset) = 1$. Prove that*

$$\sum_{S \subseteq [n]} \frac{\sigma(S)}{\pi(S)} = (n^2 + 2n) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)(n+1),$$

where the sum ranges over all subsets S of $[n] = \{1, 2, 3, \ldots, n\}$.

We can split the sum into two parts: sets S not including n, and sets S including n. The first part is subsets of $[n-1]$, and sets in the second part can be written as $T \cup \{n\}$, where T is a subset of $[n-1]$. So we have

$$\sum_{S \subseteq [n]} \frac{\sigma(S)}{\pi(S)} = \sum_{S \subseteq [n-1]} \frac{\sigma(S)}{\pi(S)} + \sum_{T \subseteq [n-1]} \frac{\sigma(T \cup \{n\})}{\pi(T \cup \{n\})} = \sum_{S \subseteq [n-1]} \frac{\sigma(S)}{\pi(S)} + \sum_{T \subseteq [n-1]} \frac{\sigma(T) + n}{\pi(T) \cdot n}.$$

We can simplify the second sum and rearrange to get

$$\sum_{S \subseteq [n]} \frac{\sigma(S)}{\pi(S)} = \left(1 + \frac{1}{n}\right) \sum_{S \subseteq [n-1]} \frac{\sigma(S)}{\pi(S)} + \sum_{T \subseteq [n-1]} \frac{1}{\pi(T)}.$$

The last summation can be solved directly: it is the product $(1+1)(1+\frac{1}{2}) \cdots (1+\frac{1}{n-1})$, since when we expand this product, each term $\frac{1}{\pi(T)}$ appears exactly once. This product telescopes to exactly n, so we get the recurrence

$$s_n = \left(1 + \frac{1}{n}\right) \cdot s_{n-1} + n,$$

where s_n is the sum we are trying to evaluate. From here, the statement we want can be shown by induction.

9. (a) *Find* $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{2^{2^k} + 1}$.

   We have

$$\frac{x}{2^x - 1} - \frac{2x}{2^{2x} - 1} = \frac{x}{2^x - 1}\left(1 - \frac{2}{2^x + 1}\right) = \frac{x}{2^x - 1} \cdot \frac{2^x - 1}{2^x + 1} = \frac{x}{2^x + 1},$$

   and in particular $\frac{2^k}{2^{2^k}+1} = \frac{2^k}{2^{2^k}-1} - \frac{2^{k+1}}{2^{2^{k+1}}-1}$. So this sum telescopes as

$$\left(\frac{2^1}{2^{2^1} - 1} - \frac{2^2}{2^{2^2} - 1}\right) + \left(\frac{2^2}{2^{2^2} - 1} - \frac{2^3}{2^{2^3} - 1}\right) + \left(\frac{2^3}{2^{2^3} - 1} - \frac{2^4}{2^{2^4} - 1}\right) + \cdots$$

   and in the end, only the first term $\frac{2^1}{2^{2^1}-1} = \frac{2}{3}$ remains.

   (b) *Show that* $\displaystyle\sum_{\text{all } k \geq 1} \frac{k}{2^k + 1} = \sum_{\text{odd } k \geq 1} \frac{k}{2^k - 1}$.

By similar logic, we have

$$\sum_{k=1}^{\infty} \frac{j \cdot 2^k}{2^{j \cdot 2^k} + 1} = \frac{j}{2^j - 1}.$$

So each term in the right-hand sum corresponds to infinitely many terms of the left-hand sum: term 1 is the sum of terms $1, 2, 4, 8, \ldots$ on the left-hand side, while term 3 is the sum of terms $3, 6, 12, 24, \ldots$ on the left-hand side, term 5 is the sum of terms $5, 10, 20, 40, \ldots$ on the left-hand side, and so on.

Since each positive integer can be uniquely factored as an odd number times a power of 2, this means that each term of the left-hand sum is covered exactly once in this way, and the two sums are equal.