2004 St. Petersburg Math Olympiad

Solutions to selected problems

(1) *Roma picked a natural number n. He chose a divisor of n, multiplied it by 4, and subtracted that result from n, getting 11. What is n? Find all possible answers and prove that there are no others.*

The two possible answers are $n = 15$ and $n = 55$.

Let d be the divisor of n chosen in the second step, so that $n = kd$. We are given $n-4d = 11$, so $11 = kd - 4d = (k-4)d$.

Since 11 is prime, it factors only as $11 \cdot 1$ and $1 \cdot 11$, so there are two possibilities: $k - 4 = 1$ and $d = 11$, or $k - 4 = 11$ and $d = 1$. In the first case, $n = kd = 5 \cdot 11 = 55$; in the second case, $n = kd = 15 \cdot 1 = 15$.

(2) *A country uses coins with values of 1, 2, 3, 5, 8, 10, 15, 20, 25, 32, 50, 57, 75, and 100 cents. A machine can trade a coin for exactly four coins of the same total value (for example, a 100 cent coin for 57, 20, 20, and 3 cent coins). By using several such exchanges, is it possible to turn a 100 cent coin into 100 1-cent coins?*

This is impossible. No coin can be traded for four 1-cent coins, since there is no coin with value 4. Thererefore after every exchange, at least one coin with value greater than 1 cent will remain.

(3) *Is it possible to fill a 5×8 grid with 1s and 3s such that in each row and column, the sum is divisible by 7?*

This is impossible.

In a column, the possible sums are $1 + 1 + 1 + 1 + 1 = 5$, $1 + 1 + 1 + 1 + 3 = 7$, $\ldots$, $3 + 3 + 3 + 3 + 3 = 15$; of these, the only value divisible by 7 is 7 itself. Therefore if the sum in each column is divisible by 7, each column contains four 1's and one 3, so there are eight 3's in the entire grid.

There are five rows, so eight 3's are not enough to put two 3's in every row. Thus, at least one row has sum either $1+1+1+1+1+1+1+1 = 8$, or $1+1+1+1+1+1+1+3 = 10$. Neither is divisible by 7. Therefore if all column sums are divisible by 7, at least one row sum will fail to be divisible by 7.

(4) *A six-digit number is called* almost lucky *if three of its digits have the same sum as the other three. (For example,* $013\,725$ *is almost lucky, since* $1 + 3 + 5 = 0 + 7 + 2$*.) Kostya buys two bus tickets with consecutive six-digit numbers, and finds that they are both almost lucky. Prove that one of them must end in* 0*.*

The key lemma in this and the following problem is that the sum of digits in an almost lucky number must be even.

To prove this, let $a_1, a_2, a_3, a_4, a_5, a_6$ be the digits of an almost lucky number. Then it satisfies an equation of the form $a_i + a_j + a_k = a_p + a_q + a_r$, so

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = a_i + a_j + a_k + a_p + a_q + a_r = 2(a_i + a_j + a_k),$$

which is even.

Now suppose n is almost lucky. If the last digit of n is not 9, then $n+1$ is obtained by adding 1 to the last digit of n, which increases the sum of its digits by 1. Since the sum of the digits of n is even, the sum of digits of $n+1$ must be odd, and $n+1$ cannot be almost lucky.

Conversely, if n and $n+1$ are both almost lucky, n must end in 9, and $n+1$ in 0.

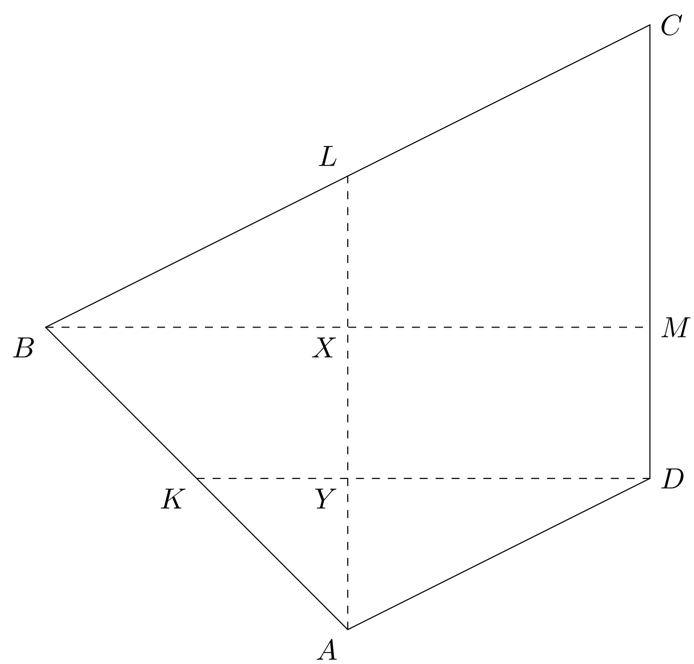
(5) *Prove that there are fewer than* 500 000 *almost lucky numbers (as defined in problem 4).*

Recall that the sum of digits of an almost lucky number is even. Pair up all 1 000 000 possible numbers by pairing n with $n+1$ if n is even, and with $n-1$ if n is odd. Of two numbers in a pair, one will have an odd sum of digits, and one will have an even sum of digits. Therefore exactly 500 000 numbers have an even sum of digits, and can be almost lucky.

However, not all of these are almost lucky: for example, 000 002 is not almost lucky, since if the digits are split into two groups, one group will sum to 2 and the other to 0. Therefore there are fewer than 500 000 almost lucky numbers.

(6) *In quadrilateral $ABCD$, let K be the midpoint of AB; let L be the midpoint of BC; let M be the point on CD such that $CM : MD = 2 : 1$. If $DK \parallel BM$ and $AL \parallel CD$, prove that $AD \parallel BC$.*

Let X be the intersection of AL and BM, and let Y be the intersection of AL and KD. One possible diagram is shown below:



As the diagram suggests, $ALCD$ is a parallelogram; we will prove this by showing that $AL = CD$ (and we already know $AL \parallel CD$).

Since $LX \parallel CM$, $\angle BLX = \angle BCM$ and $\angle BXL = \angle BMC$; therefore triangles $\triangle BLX$ and $\triangle BCM$ are similar. We are given that $BL = \frac{1}{2}BC$; therefore $LX = \frac{1}{2}CM = \frac{1}{3}CD$.

Since $XY \parallel MD$ and $XM \parallel YD$, $XYDM$ is a parallelogram; therefore $XY = MD = \frac{1}{3}CD$.

Since $BX \parallel KY$, $\angle ABX = \angle AKY$ and $\angle AXB = \angle AYK$; therefore triangles $\triangle ABX$ and $\triangle AKY$ are similar. We are given that $AK = \frac{1}{2}AB$; therefore $AY = \frac{1}{2}AX$, or $AY = YX = \frac{1}{3}CD$.

Putting these together, we conclude that $AL = AY + YX + XL = \frac{1}{3}CD + \frac{1}{3}CD + \frac{1}{3}CD = CD$.

Therefore $AL = CD$; since $AL \parallel CD$, $ALCD$ is a parallelogram, so $AD \parallel LC$, or $AD \parallel BC$, as desired.

(7) *In a group of 35 students, each is studying the same 10 subjects, and receives a numerical final grade in each: an integer from 1 to 5. (A 1 or 2 is a failing grade; 3, 4, and 5 are passing.)*

If the average grade in each subject is greater than $4\frac{2}{3}$, prove that at least 5 students did not fail any subjects.

If the average grade in a subject is greater than $4\frac{2}{3}$, then the sum of all grades is greater than $35 \cdot (4\frac{2}{3}) = 163\frac{1}{3}$.

If 4 students were to fail any given subject, the sum of their grades in that subject would be at most $2 + 2 + 2 + 2 = 8$. The remaining 31 students receive a grade of at most 5, so the sum of their grades is at most $31 \cdot 5 = 155$. The sum of the grades of all 35 students would be $8 + 155 = 163$, contradicting the claim in the previous paragraph.

Therefore at most 3 students can fail any given subject. So the total number of students that failed a subject is at most $3 \cdot 10 = 30$. There are 35 students, so at least 5 did not fail any subjects.

(8) *Real numbers $x, y > 0$ satisfy the condition $|4 - xy| < 2|x - y|$. Prove that one of x and y is less than 2 and the other is greater than 2.*

Solution 1: Square both sides of the inequality to get $(4 - xy)^2 < 4(x - y)^2$, which expands to

$$16 - 8xy + x^2y^2 < 4x^2 - 8xy + 4y^2.$$

Moving all terms to the left-hand side yields

$$x^2y^2 - 4x^2 - 4y^2 + 16 < 0,$$

which factors as $(x^2 - 4)(y^2 - 4) < 0$. For this to happen, exactly one of the factors must be negative.

Suppose it's the first factor: $x^2 - 4 < 0$ and $y^2 - 4 > 0$. Then $x^2 < 4$, so $x < 2$, while $y^2 > 4$, so $y > 2$. (Here we make use of the fact that x and y are positive.) Similarly, if the first factor were positive and the second negative, we would conclude that $x > 2$ and $y < 2$.

Solution 2: Without loss of generality, we may assume $x \geq y$. We consider three cases: $xy < 4$, $xy > 4$, and $xy = 4$. In each case, we will show $x > 2$ and $y < 2$.

First, suppose $xy < 4$. Then $4 - xy > 0$ and $x - y \geq 0$, so we have $4 - xy < 2(x - y)$, or

$$xy + 2x - 2y - 4 > 0.$$

This inequality factors as $(x - 2)(y + 2) > 0$. Since $y > 0$, $y + 2 > 0$ as well, so $x - 2 > 0$, and therefore $x > 2$. This means that $y < 4/x < 4/2 = 2$, so $x > 2$ and $y < 2$.

Next, suppose $xy > 4$. Then $4 - xy < 0$ and $x - y \geq 0$, so we have $xy - 4 < 2(x - y)$, or

$$xy - 2x + 2y - 4 < 0.$$

This inequality factors as $(x + 2)(y - 2) < 0$. Since $x > 0$, $x + 2 > 0$ as well, so $y - 2 < 0$, and therefore $y < 2$. This means that $x > 4/y > 4/2 = 2$, so $x > 2$ and $y < 2$.

Finally, suppose $xy = 4$. This almost always means that one of x and y is less than 2 and the other is greater than 2, unless $x = y = 2$. However, we are guven that $|4-xy| < 2|x-y|$, which reduces to $2(x - y) > 0$, so x is *strictly* greater than y. Therefore the two are not equal, and so $x > 2$ and $y < 2$.

(9) *A* 17×17 *grid is filled with positive numbers. In each row, the numbers form an arithmetic progression. In each column, the* squares *of the numbers form an arithmetic progression. Prove that the product of the top left and bottom right numbers is equal to the product of the numbers in the other two corners.*

Though the grid is 17×17, we can ignore all the entries. If a sequence of 17 terms is in arithmetic progression, the same will be true if we just take the 1st, 9th, and 17th term: we are simply adding 8 times more. So it suffices to take the 3×3 subgrid consisting of the four corners, the middle entries of the sides of the grid, and the center entry of the grid.

Let a, b, c, and d be the values in the corners of this 3×3 grid, giving us the array

$$\begin{bmatrix} a & ? & b \\ ? & ? & ? \\ c & ? & d \end{bmatrix}.$$

In the top row, bottom row, left column, and right column, we know two out of three terms in the arithmetic progression, so we can find the third:

$$\begin{bmatrix} a & \frac{a+b}{2} & b \\ \sqrt{\frac{a^2+c^2}{2}} & ? & \sqrt{\frac{b^2+d^2}{2}} \\ c & \frac{c+d}{2} & d \end{bmatrix}.$$

The middle entry can be found in two ways. On the one hand, it is the average of the terms to its left and right; on the other hand, its square is the average of the squares of the terms above and below it. Setting these equal, we get

$$\frac{\sqrt{\frac{a^2+c^2}{2}} + \sqrt{\frac{b^2+d^2}{2}}}{2} = \sqrt{\frac{\left(\frac{a+b}{2}\right)^2 + \left(\frac{c+d}{2}\right)^2}{2}}.$$

The remainder of the proof is just algebra. First, we can clear denominators by factoring out $\frac{1}{2\sqrt{2}}$ from both sides, getting

$$\sqrt{a^2 + c^2} + \sqrt{b^2 + d^2} = \sqrt{(a + b)^2 + (c + d)^2}.$$

Squaring both sides and expanding yields

$$(a^2 + c^2) + (b^2 + d^2) + 2\sqrt{(a^2 + c^2)(b^2 + d^2)} = (a^2 + b^2 + 2ab) + (c^2 + d^2 + 2cd),$$

or

$$\sqrt{(a^2 + c^2)(b^2 + d^2)} = ab + cd.$$

Squaring both sides and expaning again yields

$$a^2b^2 + a^2d^2 + b^2c^2 + c^2d^2 = a^2b^2 + c^2d^2 + 2abcd,$$

or

$$a^2d^2 + b^2c^2 = 2abcd.$$

Moving everything to one side, we can factor $a^2d^2 + b^2c^2 - 2abcd$ as $(ad - bc)^2$, so we get $(ad - bc)^2 = 0$. This is only possible if $ad - bc = 0$, so $ad = bc$.

(10) *In $\triangle ABC$ points K, L, and M are chosen on sides AB, BC, and AC, respectively, such that $\angle BLK = \angle CLM = \angle BAC$. Segments BM and CK intersect at P. Prove that the quadrilateral $AKPM$ is cyclic.*

We have $\triangle LMC \sim \triangle ABC$, since we are given $\angle MLC = \angle BAC$, and $\angle MCL = \angle BCA$ are the same angle. Similarly, we have $\triangle LBK \sim \triangle ABC$, since we are given $\angle BLK = \angle BAC$, and $\angle LBK = \angle ABC$ are the same angle.

Letting x be the ratio of similarity betwen $\triangle LMC$ and $\triangle ABC$, and letting y be the ratio of similarity betwen $\triangle LBK$ and $\triangle ABC$, we have $LC = x \cdot AC$, $LM = x \cdot AB$, $LB = y \cdot AB$, and $LK = y \cdot AC$.

We now show $\triangle BLM \sim \triangle KLC$. Since $\angle BLK = \angle CLM$, their supplements $\angle BLM$ and $\angle KLC$ are also equal, and we have

$$\frac{LC}{LM} = \frac{x \cdot AC}{x \cdot AB} = \frac{y \cdot AC}{y \cdot AB} = \frac{LK}{LB}.$$

Let $\alpha = \angle BAC = \angle CLM = \angle BLK$. We have $\angle MBL + \angle BML = \alpha$ by the supplementary angle theorem, and the similarity above tells us that $\angle BML = \angle KCL$, so that $\angle MBL + \angle KCL = \alpha$.

Now we finally get to introduce point P! It is the intersection of BM and CK, so $\angle MBL = \angle PBC$, and $\angle KCL = \angle PCB$. Rewriting the equation above as $\angle PBC + \angle PCB = \alpha$, we get $\angle BPC = \pi - \alpha$, so the vertical angle $\angle KPM$ is $\pi - \alpha$ as well.

Therefore $\angle KPM + \angle KAM = (\pi - \alpha) + \alpha = \pi$, which proves that $AKPM$ is cyclic.