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Probabilistic Strategies: Solutions

Western PA ARML Practice — *April 3, 2016*

1 Problems

1. *You roll two 6-sided dice. What's the probability of rolling at least one 6?*

   There is a $\frac{1}{6}$ probability of rolling a 6 on any individual die. So, there is a $\frac{2}{6} = \frac{1}{3}$ on rolling a 6 on at least 1 die out of 2. However, we could roll a 6 on both dice! We counted this possibility twice in our $\frac{1}{3}$ answer, but we should only count it once. The probability of rolling two sixes is $\frac{1}{36}$. So, the actual probability of rolling at least 1 six is $\frac{1}{3} - \frac{1}{36} = \boxed{\frac{11}{36}}$.

2. *Two teams are playing each other in a best-to-3 series. Each team has a 50% chance of winning any individual game. What's the probability that the series will go to 5 games?*

   This probability is the same as the probability that in the first four games, both teams won exactly two games. There are $\binom{4}{2} = 6$ 4-game sequences where this happens, and a total of $2^4 = 16$ possible 4-game sequences, so the probability that the series goes to 5 games is $\frac{6}{16} = \boxed{\frac{3}{8}}$.

3. *You are playing a game against a friend: you and your friend draw the top card off of a deck of cards, and you win if you and your friend both drew cards of the same suit. A deck of 52 cards is shuffled and you start playing, without shuffling the used cards back into the deck. What's the probability that you win the 5^{th} game?*

   An important observation is that the probability that you win the 5^{th} game is the same as the probaiblity you win the 1^{st} game. The probability of winning the first game is $\boxed{\frac{12}{51}}$ since after drawing one card, there are 12 cards left that match the first in suit out of a total of 51 card remaining in the deck.

4. *You roll two 6-sided dice. What's the probability that the sum of the numbers you rolled is even?*

   No matter what we rolled on the first roll, the second roll has a $\frac{1}{2}$ chance to match the parity of the first roll. The sum if even if and only if the second roll matches the parity of the first roll so the probability the sum is even is $\boxed{\frac{1}{2}}$.

5. *You're playing 3-card anti-poker, where the goal is to not make good poker hands. What's the probability that a random hand of 3 cards from a 52 card deck has cards of not all the same suit and does not contain a pair?*

   We can count the probability that a hand has cards of all the same suit or contains a pair, and take one minus that probability to find the answer. Since it is impossible to have cards of all the same suit and contain a pair, we can add the probability of having cards of all the same suit and the probability of having cards that contain a pair to find the probability that a hand has cards of all the same suit or contains a pair. The probability of having cards of all the same suit can be found by drawing cards one at a time and seeing the probability that the card matches the suit of the previous cards, so we get $\frac{12}{51} \cdot \frac{11}{50} = \frac{66}{1275}$. To find the probability that our 3 cards contains a pair, we can again find the probability that the 3 cards don't contain a pair and subtract that from 1. This gives us a probability of $1 - \frac{48}{51} \cdot \frac{44}{50}$. So, the final probability is $1 - (\frac{66}{1275} + 1 - \frac{48}{51} \cdot \frac{44}{50}) = \frac{48}{51} \cdot \frac{44}{50} - \frac{66}{1275} = \boxed{\frac{66}{85}}$.

2 More Problems

1. *(2006 AMC 10B Problem 17) Bob and Alice each have a bag that contains one ball of each of the colors blue, green, orange, red, and violet. Alice randomly selects one ball from her bag and puts it into Bob's bag. Bob then randomly selects one ball from his bag and puts it into Alice's bag. What is the probability that after this process the contents of the two bags are the same?*

   The contents of the two bags can only become the same if the ball Bob selects is the same color as the ball Alice selected. At the time of selection, Bob's bag has 6 balls, two of which have the correct color: one originally Bob's, and one originally Alice's. So the probability that Bob chooses one of them is $\frac{2}{6} = \boxed{\frac{1}{3}}$.

2. *You have a bag with 5 green balls, 2 yellow balls, and 6 red balls. You take balls out without putting them back in the bag, until there are no balls left. What's the probability that the third to last ball you draw is green or yellow?*

   For any ball, considered individually, the probability is the same. One way to see this is to think of the drawing as choosing a random permutation of balls: for example, by laying them out in order, as you draw them. Then any individual ball has an equal chance to occupy the third-to-last position. Out of 13 balls, 7 are green or yellow, so the probability is $\boxed{\frac{7}{13}}$.

3. *What's the probability that a randomly chosen integer between 1 and 1000 inclusive contains the digit 0?*

   This is most easily done by counting integers with the digit 0 in them.

There are 9 two-digit integers containing 0: $10, 20, 30, \ldots, 90$. There are 162 three-digit integers containing exactly one 0: 81 of the form $x0y$ and 81 of the form $xy0$. There are also 9 three-digit integers containing two 0's: $100, 200, 300, \ldots, 900$. Finally, there's 1000.

So the overall probability is $\frac{9+162+9+1}{1000} = \boxed{181}1000$.

4. *You're playing some games of rock-paper-scissors with a mind-reading rock-hating friend who refuses to play rock. To ensure fairness, your friend is not allowed to use his/her mind reading powers during the games, but he/she will know your strategy. What's your best strategy? (If you tie, you restart the game)*

   There is never any point to playing paper: scissors is strictly better in all cases. So your strategy must involve some probability of playing rock or scissors, and your opponent's strategy must involve some probability of playing paper or scissors.

   The case when you always play scissors is degenerate: your friend will also always play scissors, and the game will never end. However, I claim that you can do arbitrarily well by playing rock with as small probability as possible.

   Suppose you play rock with probability $\epsilon > 0$. Your opponent, knowing this, plays paper with probability p and scissors with probability $1 - p$. Then, on any given round, you have a ϵp chance of losing, and an $(1-\epsilon)p + \epsilon(1-p)$ chance of winning. Since we repeat until one or the other occurs, the overall probability of losing is

$$\frac{\epsilon p}{\epsilon p + (1-\epsilon)p + \epsilon(1-p)} = \frac{\epsilon p}{p + \epsilon - \epsilon p} = \frac{1}{1/p + 1/\epsilon - 1}.$$

   Since $\frac{1}{p} > 0$, this is at most $\frac{1}{1/\epsilon - 1}$, which can be made as small as we want by taking ϵ to be sufficiently small.

5. *You're repeatedly flipping a fair coin. You get a point if you flipped heads, and you lose a point if you flipped tails. You win if you get 10 points, and you lose if you get -5 points. What's the probability that you win?*

   Solution 1: Let p be the probability that you win (starting from 0 points). Then p is also the probability that, starting from 5 points, you lose: if you currently have 5 points, then to lose, your points have to change to -10 from your current position before they change to $+5$ from your current position, and positive and negative points are symmetric in this game.

   If you have 0 points, and we wait until you either have $+5$ or -5, then it's equally likely that we'll get to either, by symmetry. If we get to -5 first, your probability of winning is 0: you've lost. If we get to $+5$ first, your probability of winning is $1 - p$, by the above argument. So your overall probability of winning is $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot (1-p)$.

   Solving $p = \frac{1}{2}(1-p)$ for p, we get $p = \boxed{\frac{1}{3}}$.

   Solution 2: Once again, let p be the probability that you win. We compute the expected (average) number of points you have at the end of the game in two ways.

On the one hand, it must be 0: at each step, you are equally likely to gain a point and to lose a point.[1] On the other hand, it is $10p - 5(1-p)$: you have a p chance of ending with $+10$ points, and a $1-p$ chance of ending with -5 points.

Setting these equal and solving $10p - 5(1-p) = 0$, we get $p = \boxed{\dfrac{1}{3}}$.

6. *You're playing a coin flipping game against your friend. You repeatedly flip a fair coin, and you win if at any point the last two flips were Tails and then Heads. Your friend wins if the last two flips at any point are both Heads. What's the probability that you win this game?*

   Note that if the coin ever lands Tails, you've already won: the coin can keep coming up Tails for a while, but the next time it lands Heads, you will win. So the only way your friend can win is if the two first flips are both Heads, which has a probability of $\frac{1}{4}$. Therefore your probability of winning is $1 - \frac{1}{4} = \boxed{\dfrac{3}{4}}$.

   What about if you win when TTH is flipped and your friend wins when HHH is flipped?

   A similar thing happens here: if the last two flips are TT, you've won. Continue the game until the next H: the flips will then be TT$\cdots$TH, which ends in TTH. So we can change the rules to the game so that you win if we get TT before HHH.

   The relevant states of the game are T (a run of 1 tail), TT (a run of 2 tails; you win here), H (a run of 1 head), HH (a run of 2 heads), and HHH (a run of 3 heads; you lose here). We could set up some equations for the probability you win from each state, but there's an easier way.

   The second solution to problem 5 is easy to use, but we don't have point values here: so let's assign them.

   - We arbitrarily say that the state T gets $+1$ point.
   - Then H must get -1 point, so that before any coins have been flipped, when you have an equal chance of going to T or to H, your expected value is 0.
   - From T, we have an equal chance of going to TT or H. Since going to H loses 2 points, going to TT should gain 2 points, so TT must get $+3$ points.
   - From H, we have an equal chance of going to T or HH. Since going to T gains 2 points, going to HH should lose 2 points, so HH must get -3 points.
   - From HH, we have an equal chance of going to T or HHH. Since going to T gains 4 points (from -3 to $+1$), going to HHH should lose 4 points, so HHH must get -7 points.

   The game is fair at every step by design, so the expected value at the end is 0 points. However, if p is the probability that you win, then the expected value at the end is $3p - 7(1-p)$. Setting

[1] Formally, we are using a theorem called Wald's Identity here. If this seems sketchy, you are right to be worried: the identity is somewhat subtle. But the argument is sound, provided that the expected length of the game is finite, which is true here.

these equal and solving for p, we get $p = \boxed{\dfrac{7}{10}}$.

7. *(2007 AIME II Problem 10) Let S be a set with six elements. Let P be the set of all subsets of S. Subsets A and B of S, not necessarily distinct, are chosen independently and at random from P. The probability that B is contained in at least one of A or $S \setminus A$ is $\frac{m}{n^r}$, where m, n, and r are positive integers, n is prime, and m and n are relatively prime. Find $m + n + r$. (The set $S \setminus A$ is the set of all elements of S which are not in A.)*

   The probability that $B \subseteq A$ is the same as the probability that $B \subseteq S \setminus A$, since if A is equally likely to be any set in P, so is $S \setminus A$. So we begin by finding the former.

   We can think of A and B as being chosen by flipping fair coins, for each element $x \in S$, to see if $x \in A$ and then to see if $x \in B$. We have $B \subseteq A$ if, for every x, either $x \in A$ or $x \notin B$. This has a $\frac{3}{4}$ chance of happening for each $x \in S$, so overall we have $\Pr[B \subseteq A] = \frac{3^6}{4^6}$.

   We also have $\Pr[B \subseteq S \setminus A] = \frac{3^6}{4^6}$. However, both can happen at the same time: if $B = \emptyset$, then $B \subseteq A$ and $B \subseteq S \setminus A$ for any A. This has a $\frac{1}{2^6}$ chance of happening.

   So the overall probability is $\frac{3^6}{4^6} + \frac{3^6}{4^6} - \frac{1}{2^6} = \frac{697}{2^{11}}$, giving a final answer of $697 + 2 + 11 = \boxed{710}$.

8. *You're playing a game that involves scoring points by collecting cubes of the same color. Every time you receive a cube, there is a $1/2$ chance that it is white, a $1/3$ chance that it is blue, and a $1/6$ chance that it is a red cube. Your total score at the end of the game is the number of white cubes you have plus the product of the number of red and the number of blue cubes you have collected. For example, if you have 5 white cubes, 3 blue cubes, and 2 red cubes, your score would be $5 + 3 \cdot 2 = 11$ points. What is the expected value of the number of points you will have, if you receive 10 cubes over the course of the game? (The expected value of a random value is the "mean" value, for example, the expected value of a die roll is 3.5.)*

   Let W, B, and R be the number of white, blue, and red cubes drawn. We can separately compute the expected value of W and of BR.

   The expected value of W is easy: each cube has a $\frac{1}{2}$ chance of contributing a point to W, so the average contribution of all ten cubes is $\frac{1}{2} \cdot 10 = 5$.

   To compute the expected value of BR, we think of this number as counting the ordered pairs (i, j) such that the i^{th} cube is blue and the j^{th} cube is red. There are 100 such pairs, but 10 of them are pairs (i, i), which can't possibly be the right combination of colors. Of the remaining 90, each pair has a $\frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$ chance of being the right pair of colors, for an overall expected value of $90 \cdot \frac{1}{18} = 5$.

   Putting these together, we conclude that the overall expected value is $5 + 5 = \boxed{10}$.