# POWER ROUND: MEDITATIONS ON PARTITIONS

ARML 1998

(1) *Let positive integers $A$, $B$, and $C$ be the angles of a triangle (in degrees) such that $A \leq B \leq C$.*

(a) *Determine all the values that each of $A$, $B$, and $C$ can take on.*

The first angle, $A$, can be any integer $1 \leq A \leq 60$. Since $A \leq B \leq C$, $180 = A+B+C \geq A+A+A$, so $A \leq 60$; we are given the lower bound. We also check that setting $B = A$ and $C = 180 - 2A$ lets us get any of these values.

The second angle, $B$, can be any integer $1 \leq B \leq 89$. Since $B \leq C$, $180 = A+B+C \geq 1 + B + B$, so $B \leq 89.5$, and since $B$ is an integer, it can be at most 89; we are given the lower bound. The examples above give us values of $B$ between 1 and 60; for $45 \leq B \leq 89$, take a right triangle with angles $A = 90 - B$ and $C = 90$.

The final angle, $C$, can be any integer $60 \leq C \leq 178$. For the lower bound: $180 = A+B+C \leq C+C+C$, so $C \geq 60$. For the upper bound: $180 = A+B+C \geq 1+1+C$, so $C \leq 178$. For any of these, we can set $B = 60$ and $A = 120 - C$ to obtain a valid triangle.

(b) *Compute the number of ordered triples $(A, B, C)$ in which $B = 70°$.*

The answer is 40.

Since $B \leq C$, we have $C \geq 70$, so $A = 180 - B - C \leq 180 - 140 = 40$. We then check that any value of $A$ between 1 and 40 works: the ordered triples

$$(1, 70, 109), \quad (2, 70, 108), \quad \ldots, \quad (40, 70, 70)$$

all correspond to valid triangles.

Note: it is not hard to show that any three positive integers that add to 180 can be angles of a triangle. Given such $A$, $B$, and $C$, divide the perimeter of a circle by three points into arcs of measure $2A$, $2B$, and $2C$, which are in total 360. Then take the triangle with those three points as vertices.

(2) *In convex pentagon $ABCDE$, $\text{m}\angle A < \text{m}\angle B < \text{m}\angle C < \text{m}\angle D < \text{m}\angle E$. Let $T = \text{m}\angle C + \text{m}\angle D$. If $\text{m}\angle A : \text{m}\angle B : \text{m}\angle C : \text{m}\angle D : \text{m}\angle E = 1 : 2 : x : y : 5$, determine the range of values of $T$.*

If we had the non-strict inequality on the angles, we could put them in the ratio $1 : 2 : 2 : 2 : 5$, making the angles $45°, 90°, 90°, 90°$, and $225°$ and $T = 180°$. We could also put them in the ratio $1 : 2 : 5 : 5 : 5$, making the angles $30°, 60°, 150°, 150°$, and $150°$ and $T = 300°$. Any value between these is achievable.

Since the angles must be strictly increasing, these endpoints are ruled out, but we still can get any $180 < T < 300$.

(3) *Let $a$, $b$, and $c$ be positive integers such that $a < 3b$ and $b > 4c$ and $a + b + c = 200$.*

(a) *Determine the largest value that $c$ can take on.*

The answer is 39. We can get this with $a = 4$, $b = 157$, $c = 39$, which satisfies all the inequalities.

Since $a > 1$ and $b > 4c$, we have $a+b+c > 1+4c+c = 5c+1$, but $a+b+c = 200$. So $5c+1 < 200$, which means $c < 39.8$. Since $c$ is an integer, $c \leq 39$, so no larger value is possible.

(b) *Determine the smallest value that $b$ can take on.*

The answer is 48. We can get this with $a = 141$, $b = 48$, and $c = 11$, which satisfies all the inequalities.

Since $b > 4c$, we have $c < \frac{b}{4}$, and we already knew $a < 3b$, which means $a + b + c < 3b + b + \frac{b}{4} = \frac{17}{4}b$. Since $a + b + c = 200$, we have $\frac{17}{4}b > 200$, so $b > \frac{800}{17} \approx 47.06$. Since $b$ is an integer, $b \geq 48$, so no smaller value is possible.

(c) *Determine the number of ordered triples $(a, b, c)$ in which $c = 11$.*

There are 141 such triples.

Setting $b$ to any value $48 \leq b \leq 188$ will work. We must then have $a = 189 - b$, which is always a positive integer less than $3b$, and $b$ is always at greater than $4c = 44$. But $b \geq 189$ will not work (since $a$ becomes 0 or less) and $b < 48$ was shown impossible in part (b).

(4) *Let $a$, $b$, and $c$ be positive integers. If $a + b + c = 85$, $c > 3a$, $2b > c$, and $5a > 3b$, prove algebraically that there is a unique solution $(a, b, c)$ to this system.*

Solving the equations for $a$, we get $\frac{3}{2}a < b < \frac{5}{3}a$, and $3a < c < \frac{10}{3}a$.

If we plug the lower bounds into $a + b + c = 85$, we get $\frac{11}{2}a < 85$, so $a < \frac{170}{11}$, which means $a \leq 15$. Plugging the upper bounds into the same equation, we get $6a > 85$, so $a \geq 14$.

Now if we try $a = 14$, we have $c > 42$ (which means $c \geq 43$), $2b > c$ (which means $b \geq 22$), and $5a > 3b$ (which means $b \leq 23$).

If we take $a = 14$, then $5a > 3b$ tells us $b \leq 23$, which means $c = 85 - a - b \geq 48$. However, $c \geq 48$ and $b \leq 23$ violates $2b > c$, so this is impossible.

If we take $a = 15$, then $5a > 3b$ tells us $b \leq 24$. In fact we must have $b = 24$, because $b \leq 23$ would give us $c \geq 47$, with the same problems as before. If $a = 15$ and $b = 24$, then $c = 46$, and we can check that $(15, 24, 46)$ satisfies all the equations.

(5) *A unit square is divided into 4 rectangles of positive area by two cuts parallel to the sides of the square. Let $a_1 \leq a_2 \leq a_3 \leq a_4$ be the areas of the four parts in nondecreasing order. For each $i = 1, \ldots, 4$, determine with proof the range of values for $a_i$.*

The answer is that $0 < a_1 \leq \frac{1}{4}$, $0 < a_2 \leq \frac{1}{4}$, $0 < a_3 < \frac{1}{2}$, and $\frac{1}{4} \leq a_4 < 1$.

To show the bounds on $a_1$ and $a_4$, just note that $a_1$ (the smallest area) can't be larger than the average area $\frac{a_1+a_2+a_3+a_4}{4} = \frac{1}{4}$, while $a_4$ (the largest area) can't be less than the average area.

To show the bound on $a_2$, we have to work harder. Suppose one cut is at distance $x$ from one of its parallel sides, and the other cut is at distance $y$ from one of its parallel sides, with

$0 < x \le y < \frac{1}{2}$. Then $a_1 = xy$, $a_2 = x(1-y)$, $a_3 = (1-x)y$, and $a_4 = (1-x)(1-y)$. Since $x \le y$, we have $a_2 = x(1-y) \le x(1-x) = \frac{1}{4} - (x - \frac{1}{2})^2$, which can be at most $\frac{1}{4}$.

To show the bound on $a_3$, note that $a_1 \le a_2$ and $a_3 \le a_4$, so $a_1 + a_3 \le a_2 + a_4$, which means $a_1 + a_3 \le \frac{1}{2}$. Since $a_1 > 0$, we have $a_3 < \frac{1}{2}$.

To show that all of these bounds are best possible, consider the following three dissections:

- Both cuts divide the square equally, giving $a_1 = a_2 = a_3 = a_4 = \frac{1}{4}$.
- Both cuts are within some small distance $\epsilon$ of a side, giving $a_1 = \epsilon^2$, $a_2 = a_3 = \epsilon(1-\epsilon)$, and $a_4 = (1-\epsilon)^2$. By taking $\epsilon$ arbitrarily small, $a_1$, $a_2$, and $a_3$ get arbitrarily close to 0 and $a_4$ gets arbitrarily close to 1.
- One cut divides the square equally while the other is within $\epsilon$ of a side, giving $a_1 = a_2 = \frac{1}{2}\epsilon$ and $a_3 = a_4 = \frac{1}{2}(1-\epsilon)$. By taking $\epsilon$ arbitrarily small, $a_3$ gets arbitrarily close to $\frac{1}{2}$.

(6) A unit cube is divided into 8 parallelepipeds of positive volume by three cuts parallel to the faces of the cube. Let $v_1 \le v_2 \le \cdots \le v_8$ be the volumes of the eight parts in nondecreasing order. Determine with proof the range of values for $v_4$ and $v_5$.

The ranges are $0 < v_4 \le \frac{1}{8}$ and $0 < v_5 < \frac{1}{4}$. The lower bounds here are trivial.

To show the upper bound on $v_5$, note that $v_5 + v_6 + v_7 + v_8 < 1$, and $v_5$ is the smallest of these, so $v_5 < \frac{1}{4}$.

To show the upper bound on $v_4$, we have to do some tedious work. Suppose that the cuts are made at distances $x$, $y$, and $z$ from a parallel face, with $0 < x \le y \le z \le \frac{1}{2}$. Then the eight volumes are products like $xyz$ or $(1-x)y(1-z)$, and we can say the following things about their order:

- $xyz \le xy(1-z) \le x(1-y)z \le (1-x)yz$.
- $x(1-y)(1-z) \le (1-x)y(1-z) \le (1-x)(1-y)z \le (1-x)(1-y)(1-z)$.
- $x(1-y)z \le x(1-y)(1-z)$.

So $v_4$ is the smaller of $(1-x)yz$ and $x(1-y)(1-z)$, and $v_5$ is the larger.

We have $v_4 v_5 = x(1-x)y(1-y)z(1-z)$; also, $x(1-x) \le \frac{1}{4}$ (as shown in the previous problem), $y(1-y) \le \frac{1}{4}$, and $z(1-z) \le \frac{1}{4}$. Therefore $v_4 v_5 \le \frac{1}{64}$. Since $v_4 \le v_5$, we must have $v_4 \le \frac{1}{8}$.

To show that these bounds are best possible, consider the following examples:

- All three cuts are even, so all eight volumes are $\frac{1}{8}$.
- Two cuts are even, and the third is arbitrarily close to a face. Then $v_1$ through $v_4$ will be arbitrarily close to 0, and $v_5$ through $v_8$ arbitrarily close to $\frac{1}{4}$.
- All three cuts are arbitrarily close to one of the faces they're parallel to. Then $v_1$ through $v_7$ will be arbitrarily close to 0, and $v_8$ arbitrarily close to 1.

(7) *Let $n$ be a positive integer. Allie and Bob play a game constructing a partition $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \ge a_2 \ge \cdots \ge a_k \ge 1$. Allie wins if there is an odd number of terms in the partition, i.e. if $k$ is odd, and Bob wins otherwise. Allie begins by choosing an $a_1$ between 1 and $n-1$ inclusive. Bob then chooses an $a_2$ between 1 and $a_1$ inclusive such that*

*$a_1 + a_2 \le n$. Allie then chooses an $a_3$ between 1 and $a_2$ inclusive such that $a_1 + a_2 + a_3 \le n$, and so on, with the game ending when the partition is complete. Determine with proof all $n > 1$ for which Bob has a winning strategy.*

Bob has a winning strategy for $n$ if and only if $n$ is a power of 2.

All we need to keep track of over the course of the game is the *limit* $\ell$ (initially $\ell = n - 1$) that is the largest number you can write down, and the *remainder* $r$ (initally $r = n$) equal to the difference between $n$ and the sum of all numbers written. Writing down a number $a$ changes the limit $\ell$ to $a$ and the remainder $r$ to $r - a$. The player who gets $r$ down to 0 wins.

The winning strategy in this game is to try, on your turn, to achieve a position $(\ell, r)$ such that, for some $i$, $\ell < 2^i$ and $r$ is divisible by $2^i$. We call such a position *$i$-uncomfortable*, with the idea that your goal is to place your opponent in an uncomfortable position.

To prove this strategy, we check the following three facts:

- From an $i$-uncomfortable position, your opponent can't win in one turn. Either $r = 0$ (and you've already won), or $r \ge 2^i$ (and no number that's at most $\ell$ can reduce $r$ to 0).
- Moreover, from an $i$-uncomfortable position, your opponent can't produce another uncomfortable position.
- However, from any comfortable position, you can place your component in an $i$-uncomfortable position for some $i$.

So if the "make your opponent uncomfortable" strategy is executed, your position will never be uncomfortable, and your opponent's position will always be. Eventually $r$ will get down to 0 and someone will win: that will have to be you, because your opponent can't win from an uncomfortable position.

To show the second claim, suppose that position $(\ell, r)$ is $i$-uncomfortable, so $r$ is divisible by $2^i$ and $\ell < 2^i$. No move below the limit can produce another multiple of $2^i$. To produce a multiple of $2^{i-1}$, you need to subtract at least $2^{i-1}$ from $r$. To produce a multiple of $2^{i-2}$, you need to subtract at least $2^{i-2}$, and so on. So the new limit will be at least as big as the largest power of 2 dividing $r$, and the new position is comfortable.

To show the third claim, let $(\ell, r)$ be comfortable; let $2^i$ be the largest power of 2 less than $\ell$. Since $(\ell, r)$ is not $i$-uncomfortable, $r$ is not divisible by $2^i$. So let $a = r \bmod 2^i$ be the next move. Then the new limit, $a$, is less than $2^i$, and the new remainder, $r - a$, is divisible by $2^i$, so we've produced an $i$-uncomfortable position.

If the starting position $(n - 1, n)$ is comfortable, then Allie can execute this strategy and win. This happens most of the time; however, when $n = 2^i$ for some $i$, $(n - 1, n)$ is $i$-uncomfortable. So after Allie's first move, Bob will be an a comfortable position, and can execute this strategy to win.

(8) *Allie and Bob play a game similar to the one in (7) except that the inequality $a_i \ge a_{i+1}$ is replaced by $2a_i \ge a_{i+1}$. Prove that Bob has a winning strategy if and only if $n$ is a Fibonacci number. (You may assume the following: each positive integer $n$ can be uniquely represented as a decreasing sum of non-adjacent Fibonacci numbers, e.g., $32 = 21 + 8 + 3$.)*

The idea here is similar, but the new constraint changes the possible moves: from a position with limit $\ell$ and remainder $r$, one may choose a number $a$ with $1 \leq a \leq \ell$ and pass to the position with limit $2a$ and remainder $r - a$.

We correspondingly define a new way to tell if a position is comfortable. We say that $(\ell, r)$ is comfortable if, when $r$ is written as a decreasing sum of non-adjacent Fibonacci numbers, the smallest number is at most $\ell$.

To prove that this lets us implement a "make your opponent uncomfortable strategy", we check two things:

- From a comfortable position, an uncomfortable one can always be produced.
- From an uncomfortable position, any move leads to a comfortable one (and no move can win).

We begin with the first claim. In the position $(\ell, r)$, let $r$ have the non-adjacent Fibonacci representation of $F_{i_1} + F_{i_2} + \cdots + F_{i_j}$, where $F_{i_j}$ is the smallest. If the position is comfortable, $a = F_{i_j}$ can be written down. Then the new limit is $2a = 2F_{i_j}$, and the new remainder is $r - a = F_{i_1} + F_{i_2} + \cdots + F_{i_{j-1}}$. But $F_{i_{j-1}} \geq F_{i_j+2} = F_{i_j} + F_{i_j+1} > F_{i_j} + F_{i_j}$, so it's greater than the new limit. So the new position is uncomfortable.

Next, we show the second claim. Suppose $(\ell, r)$ is an uncomfortable position, with $r = F_{i_1} + F_{i_2} + \cdots + F_{i_j}$. The next move is some number less than $F_{i_j}$, since $\ell < F_{i_j}$. So the Fibonacci representation of the next remainder will have the same initial segment, with merely $F_{i_j}$ replaced by some smaller Fibonacci numbers.

Thus, we may ignore this unchanging beginning, and assume that $r = F_i$ for some $i$, and $\ell < F_i$. Suppose there existed a move to another uncomfortable position $(2a, r - a)$. Then the non-adjacent Fibonacci representations for $r - a$ and for $2a$ could be concatenated, since the last Fibonacci number in the representation of $r - a$ is greater than $2a$, so it can't be adjacent to the first Fibonacci number in the representation of $a$. This would give us a second representation for $r = F_i$, contradicting the uniqueness the problem lets us assume.

So the "make your opponent uncomfortable" strategy is a viable one, and can be summarized as follows:

(a) Write the current remainder $r$ as a sum of non-adjacent Fibonacci numbers.

(b) If the smallest Fibonacci number is playable, write it down.

(c) If not, you're in a position with no winning strategy: your opponent can win with optimal play.

The starting position in this game has a remainder of $n$ and a limit of $n - 1$. The only way this can be uncomfortable is if $n$ is a Fibonacci number; if $n$ is a sum of two or more non-adjacent Fibonacci numbers, the smaller of them will be less than $n$, so below the limit. Therefore Allie wins games starting from non-Fibonacci numbers, with optimal play, and Bob wins games starting from Fibonacci numbers.