Mock ARML

Individual Round

ARML Practice 3/27/2016

Warm-up problems

(ARML 1992) For a positive integer n, $n(n+1)(n+2)(n+3)(n+4)$ is divisible by both 13 and 31. Find the smallest possible value of n.

Find all pairs of real numbers a and b such that $a+b$ is an integer and $a^2+b^2=2$.

Warm-up problems

(ARML 1992) For a positive integer n, $n(n+1)(n+2)(n+3)(n+4)$ is divisible by both 13 and 31. Find the smallest possible value of n.

Answer: $n = 61$.

Find all pairs of real numbers a and b such that $a+b$ is an integer and $a^2+b^2=2$.

Answer: $(a,b)=(\pm 1,\pm 1)$, $(a,b)=(\frac{1}{2}\pm\frac{\sqrt{3}}{2},\frac{1}{2}\mp\frac{\sqrt{3}}{2})$, and $(a,b)=(-\frac{1}{2}\pm\frac{\sqrt{3}}{2},-\frac{1}{2}\mp\frac{\sqrt{3}}{2})$

Problems 1 and 2

1. Given an integer n, let $S(n)$ denote the sum of the digits of n. Compute the largest 3-digit number N such that $S(N) = 2S(2N)$.

2. The formula $F = \frac{9}{5}C + 32$ converts Celsius (C) temperature into Fahrenheit (F). Find the set of temperatures (in Celsius) for which F is between $\frac{1}{2}C$ and $2C$.

Problems 1 and 2

1. Given an integer n, let $S(n)$ denote the sum of the digits of n. Compute the largest 3-digit number N such that $S(N) = 2S(2N)$.

Answer: 855

2. The formula $F = \frac{9}{5}C + 32$ converts Celsius (C) temperature into Fahrenheit (F). Find the set of temperatures (in Celsius) for which F is between $\frac{1}{2}C$ and $2C$.

Answer: $\boldsymbol{C \geq 160}$ **or** $\boldsymbol{C \leq -\frac{320}{13}}$

Problems 3 and 4

3. Compute the integer closest to

$$\log_2 \frac{2+2^2+2^{2^2}+2^{2^{2^2}}+2^{2^{2^{2^2}}}}{2+2^2+2^{2^2}+2^{2^{2^2}}}.$$

4. Multiplying together the areas of an equilateral triangle with side x, a square with side x, and a regular hexagon with side x yields y. Compute the smallest integer $y > 2016$ for which x will also be an integer.

Problems 3 and 4

3. Compute the integer closest to

$$\log_2 \frac{2 + 2^2 + 2^{2^2} + 2^{2^{2^2}} + 2^{2^{2^{2^2}}}}{2 + 2^2 + 2^{2^2} + 2^{2^{2^2}}}.$$

Answer: 65 520

4. Multiplying together the areas of an equilateral triangle with side x, a square with side x, and a regular hexagon with side x yields y. Compute the smallest integer $y > 2016$ for which x will also be an integer.

Answer: 4608

Problems 5 and 6

5. (1993) Two of the diagonals of a convex equilateral pentagon are perpendicular. If one of the interior angles of the pentagon is 100°, compute the measures of the other four interior angles.

6. Liouville's constant

$$L = 0.11000100000000000000000100\ldots$$

is defined to have a 1 in the n^{th} place after the decimal if $n = k!$ for some k, and 0 otherwise.

Compute the sum of the first 2016 digits of L^2 after the decimal.

Problems 5 and 6

5. (1993) Two of the diagonals of a convex equilateral pentagon are perpendicular. If one of the interior angles of the pentagon is 100°, compute the measures of the other four interior angles.

Answer: 60, 80, 140, and 160.

6. Liouville's constant

$$L = 0.11000100000000000000000100\ldots$$

is defined to have a 1 in the n^{th} place after the decimal if $n = k!$ for some k, and 0 otherwise.

Compute the sum of the first 2016 digits of L^2 after the decimal.

Answer: 36

Problems 7 and 8

7. (2000) If the last 7 digits of $n!$ are $8\,000\,000$, compute n.

8. A function $f : \{2, \dots, N\} \to [0, \infty)$ satisfies the equation $f(xy + 1) = f(x) + f(y) + 1$ for all integers $x, y \geq 2$. Compute the largest possible value of N.

Problems 7 and 8

7. (2000) If the last 7 digits of $n!$ are $8\,000\,000$, compute n.

Answer: 27

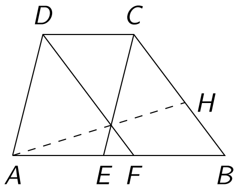
8. A function $f : \{2, \ldots, N\} \to [0, \infty)$ satisfies the equation $f(xy + 1) = f(x) + f(y) + 1$ for all integers $x, y \geq 2$. Compute the largest possible value of N.

Answer: 32

9. (1986) Compute

$$\frac{(1+17)(1+\frac{17}{2})(1+\frac{17}{3})\cdots(1+\frac{17}{19})}{(1+19)(1+\frac{19}{2})(1+\frac{19}{3})\cdots(1+\frac{19}{17})}.$$

10. (2015) In trapezoid $ABCD$ with bases AB and CD, $AB = 14$ and $CD = 6$. Points E and F lie on AB such that $AD \parallel CE$ and $BC \parallel DF$. Segments DF and CE intersect at G, and AG intersects BC at H. Compute $\frac{[CGH]}{[ABCD]}$.



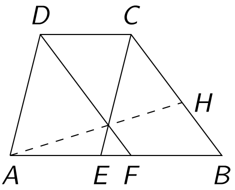
Problems 9 and 10

9. (1986) Compute

$$\frac{(1+17)(1+\frac{17}{2})(1+\frac{17}{3})\cdots(1+\frac{17}{19})}{(1+19)(1+\frac{19}{2})(1+\frac{19}{3})\cdots(1+\frac{19}{17})}.$$

Answer: 1

10. (2015) In trapezoid $ABCD$ with bases AB and CD, $AB = 14$ and $CD = 6$. Points E and F lie on AB such that $AD \parallel CE$ and $BC \parallel DF$. Segments DF and CE intersect at G, and AG intersects BC at H. Compute $\frac{[CGH]}{[ABCD]}$.



Answer: $\mathbf{\frac{27}{160}}$