Complex Numbers Solutions

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Solutions

1. (2009 AIME I Problem 2) There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z+n} = 4i.$$

Find n.

[Solution: $n = 697$]

$$\frac{z}{z+n} = 4i \implies 1 - \frac{n}{z+n} = 4i \implies 1 - 4i = \frac{n}{z+n} \implies \frac{1}{1-4i} = \frac{z+n}{n}$$

$$\implies \frac{1+4i}{17} = \frac{z}{n} + 1$$

Since their imaginary part has to be equal,

$$\frac{4i}{17} = \frac{164i}{n} \implies n = \frac{(164)(17)}{4} = 697 \implies n = \boxed{697}.$$

2. (1985 AIME Problem 3) Find c if a, b, and c are positive integers which satisfy $c = (a+bi)^3 - 107i$, where $i^2 = -1$.

[Solution: $c = 198$, where $a = 6$ and $b = 1$]

Expanding out both sides of the given equation we have $c + 107i = (a^3 - 3ab^2) + (3a^2b - b^3)i$. Two complex numbers are equal if and only if their real parts and imaginary parts are equal, so $c = a^3 - 3ab^2$ and $107 = 3a^2b - b^3 = (3a^2 - b^2)b$. Since a, b are integers, this means b is a divisor of 107, which is a prime number. Thus either $b = 1$ or $b = 107$. If $b = 107$, $3a^2 - 107^2 = 1$ so $3a^2 = 107^2 + 1$, but $107^2 + 1$ is not divisible by 3, a contradiction. Thus we must have $b = 1$, $3a^2 = 108$ so $a^2 = 36$ and $a = 6$ (since we know a is positive). Thus $c = 6^3 - 3 \cdot 6 = \boxed{198}$.

3. (1995 AIME Problem 5) For certain real values of $a, b, c,$ and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i = \sqrt{-1}$. Find b.

[Solution: $b = 051$]

Since the coefficients of the polynomial are real, it follows that the non-real roots must come in complex conjugate pairs. Let the first two roots be m, n. Since $m + n$ is not real, m, n are not conjugates, so the other pair of roots must be the conjugates of m, n. Let m' be the conjugate of m, and n' be the conjugate of n. Then,

$$m \cdot n = 13 + i, m' + n' = 3 + 4i \implies m' \cdot n' = 13 - i, m + n = 3 - 4i.$$

By Vieta's formulas, we have that $b = mm' + nn' + mn' + nm' + mn + m'n' = (m+n)(m' + n') + mn + m'n' = \boxed{051}$.

4. (1984 AIME Problem 8) The equation $z^6 + z^3 + 1 = 0$ has complex roots with argument θ between $90°$ and $180°$ in the complex plane. Determine the degree measure of θ.

   [Solution: $\theta = 160°$]

   We shall introduce another factor to make the equation easier to solve. If r is a root of $z^6 + z^3 + 1$, then $0 = (r^3 - 1)(r^6 + r^3 + 1) = r^9 - 1$. Thus, the root we want is also a 9th root of unity.

   This reduces θ to either $120°$ or $160°$. But θ can't be $120°$ because if $r = \cos 120° + i \sin 120°$, then $r^6 + r^3 + 1 = 3$. This leaves $\boxed{\theta = 160}$.

5. (1994 AIME Problem 8) The points $(0,0)$, $(a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find the value of ab.

   [Solution: $ab = 315$]

   Consider the points on the complex plane. The point $b+37i$ is then a rotation by $60°$ of $a+11i$ about the origin, so

$$(a+11i)(\text{cis}\, 60°) = (a+11i)\left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right) = b + 37i.$$

   Equating the real and imaginary parts, we have:

$$b = \frac{a}{2} - \frac{11\sqrt{3}}{2} \implies 37 = \frac{11}{2} + \frac{a\sqrt{3}}{2} \implies a = 21\sqrt{3} \implies b = 5\sqrt{3}$$

   Thus, the answer is $ab = (21\sqrt{3})(5\sqrt{3}) = \boxed{315}$.

   Note: There is another solution where the point $b + 37i$ is a rotation of -60 degrees of $a + 11i$; however, this triangle is just a reflection of the first triangle by the y-axis, and the signs of a and b are flipped. However, the product ab is unchanged.

6. (1999 AIME Problem 9) A function f is defined on the complex numbers by $f(z) = (a + bi)z$, where a and b are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a+bi| = 8$ and that $b^2 = m/n$, where m and n are relatively prime positive integers, find $m + n$.

   [Solution: $m + n = 259$, where $m = 255$ and $n = 4$]

   Plugging in $z = 1$ yields $f(1) = a + bi$. This implies that $a + bi$ must fall on the line $Re(z) = \frac{1}{2} \implies a = \frac{1}{2}$, given the equidistant rule. By $|a + bi| = 8$, we get $a^2 + b^2 = 64$, and plugging in $a = \frac{1}{2}$ yields $b^2 = \frac{255}{4}$. The answer is thus $\boxed{259}$.

7. (2000 AIME II Problem 9) Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3°$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

   [Solution: 000]

   Using the quadratic equation on $z^2 - (2\cos 3)z + 1 = 0$, we have $z = \frac{2\cos 3 \pm \sqrt{4\cos^2 3 - 4}}{2} = \cos 3 \pm i \sin 3 = \text{cis}\, 3°$.

   Using De Moivre's Theorem we have $z^{2000} = \cos 6000° + i \sin 6000°$, $6000 = 16(360) + 240$, so $z^{2000} = \cos 240° + i \sin 240°$.

   We want $z^{2000} + \frac{1}{z^{2000}} = 2\cos 240° = -1$. Finally, the least integer greater than -1 is $\boxed{000}$.

8. (2005 AIME II Problem 9) For how many positive integers $n \leq 1000$ is $(\sin t + i \cos t)^n = \sin nt + i \cos nt$ true for all real t?

   [Solution: 250, where $n \in 1 + 4\mathbb{Z}$]

   This problem begs us to use the familiar identity $e^{it} = \cos(t) + i\sin(t)$. Notice that $\sin(t) + i\cos(t) = i(\cos(t) - i\sin(t)) = ie^{-it}$ since $\sin(-t) = -\sin(t)$. Using this, $(\sin(t) + i\cos(t))^n = \sin(nt) + i\cos(nt)$ is recast as $(ie^{-it})^n = ie^{-itn}$. Hence we must have $i^n = i \Rightarrow i^{n-1} = 1 \Rightarrow n \equiv 1 \bmod 4$. Thus since 1000 is a multiple of 4 exactly one quarter of the residues are congruent to 1 hence we have $\boxed{250}$.

9. (1990 AIME Problem 10) The sets $A = \{z : z^{18} = 1\}$ and $B = \{w : w^{48} = 1\}$ are both sets of complex roots of unity. The set $C = \{zw : z \in A \text{ and } w \in B\}$ is also a set of complex roots of unity. How many distinct elements are in C?

   [Solution: $|C| = 144$]

   The least common multiple of 18 and 48 is 144, so define $n = e^{2\pi i/144}$. We can write the numbers of set A as $\{n^8, n^{16}, \ldots n^{144}\}$ and of set B as $\{n^3, n^6, \ldots n^{144}\}$. n^x can yield at most 144 different values. All solutions for zw will be in the form of $n^{8k_1+3k_2}$.

   8 and 3 are relatively prime, and it is well known that for two relatively prime integers a and b, the largest number that cannot be expressed as the sum of multiples of a and b is $(ab - a - b)$. For $3, 8$, this is 13; however, we can easily see that the numbers 145 to 157 can be written in terms of 3 and 8. Since the exponents are of roots of unities, they reduce $\bmod 144$, so all numbers in the range are covered. Thus the answer is $\boxed{144}$.

10. (1992 AIME Problem 10) Consider the region A in the complex plane that consists of all points z such that both $\frac{z}{40}$ and $\frac{40}{\bar{z}}$ have real and imaginary parts between 0 and 1, inclusive. What is the integer that is nearest the area of A?

    [Solution: $[A] \approx 572$]

    Let $z = a + bi \implies \frac{z}{40} = \frac{a}{40} + \frac{b}{40}i$. Since $0 \leq \frac{a}{40}, \frac{b}{40} \leq 1$ we have the inequality

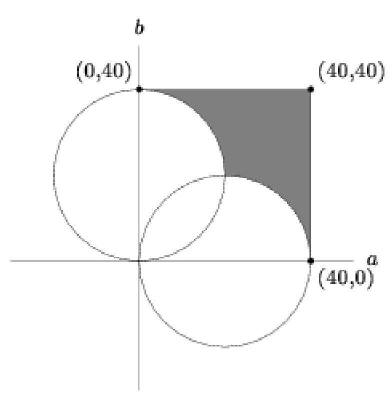
    $$0 \leq a, b \leq 40$$

    which is a square of side length 40.

    Also, $\frac{40}{\bar{z}} = \frac{40}{a-bi} = \frac{40a}{a^2+b^2} + \frac{40b}{a^2+b^2}i$ so we have $0 \leq a, b \leq \frac{a^2+b^2}{40}$, which leads to

    $$(a - 20)^2 + b^2 \geq 20^2$$
    $$a^2 + (b - 20)^2 \geq 20^2$$

    We graph them:



We want the area outside the two circles but inside the square. Doing a little geometry, the area of the intersection of those three graphs is $40^2 - \frac{40^2}{4} - \frac{1}{2}\pi 20^2 = 1200 - 200\pi \approx 571.68$. Thus, by rounding to the nearest integer we get $\boxed{572}$.

11. (1988 AIME Problem 11) Let $w_1, w_2, \ldots, w_n$ be complex numbers. A line L in the complex plane is called a mean line for the points $w_1, w_2, \ldots, w_n$ if L contains points (complex numbers) $z_1, z_2, \ldots, z_n$ such that

$$\sum_{k=1}^{n}(z_k - w_k) = 0.$$

For the numbers $w_1 = 32 + 170i$, $w_2 = -7 + 64i$, $w_3 = -9 + 200i$, $w_4 = 1 + 27i$, and $w_5 = -14 + 43i$, there is a unique mean line with y-intercept 3. Find the slope of this mean line.

[Solution: 163]

$$\sum_{k=1}^{5} z_k - \sum_{k=1}^{5} w_k = 0 \implies \sum_{k=1}^{5} z_k = 3 + 504i$$

Each $z_k = x_k + y_k i$ lies on the complex line $y = mx + 3$, so we can rewrite this as

$$\sum_{k=1}^{5} z_k = \sum_{k=1}^{5} x_k + \sum_{k=1}^{n} y_k i \implies 3 + 504i = \sum_{k=1}^{5} x_k + i\sum_{k=1}^{5}(mx_k + 3)$$

Matching the real parts and the imaginary parts, we get that $\sum_{k=1}^{5} x_k = 3$ and $\sum_{k=1}^{5}(mx_k + 3) = 504$. Simplifying the second summation, we find that $m\sum_{k=1}^{5} x_k = 504 - 3 \cdot 5 = 489$, and substituting, the answer is $m \cdot 3 = 489 \implies m = \boxed{163}$.

12. (1996 AIME Problem 11) Let P be the product of the roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have a positive imaginary part, and suppose that $\mathrm{P} = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $0 < r$ and $0 \le \theta < 360$. Find θ.

[Solution:]

Let $w =$ the 5th roots of unity, except for 1. Then $w^6+w^4+w^3+w^2+1 = w^4+w^3+w^2+w+1 = 0$, and since both sides have the fifth roots of unity as roots, we have that $z^4 + z^3 + z^2 + z + 1 | z^6 + z^4 + z^3 + z^2 + 1$. Long division quickly gives the other factor to be $z^2 - z + 1$. Thus, $z^2 - z + 1 = 0 \implies z = \frac{1\pm\sqrt{-3}}{2} = \text{cis } 60, \text{cis } 300$

Discarding the roots with negative imaginary parts (leaving us with $\text{cis}\theta,\ 0 < \theta < 180$), we are left with cis $60, 72, 144$; their product is $P = \text{cis}(60 + 72 + 144) = \text{cis}\boxed{276}$.

13. (1997 AIME Problem 11)

Let $x = \dfrac{\sum\limits_{n=1}^{44} \cos n^\circ}{\sum\limits_{n=1}^{44} \sin n^\circ}$. What is the greatest integer that does not exceed $100x$?

[Solution: 241]

Using the identity $\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2} \implies \sin x + \cos x = \sin x + \sin(90 - x) = 2\sin 45\cos(45 - x) = \sqrt{2}\cos(45 - x)$, note that

$$\sum_{n=1}^{44}\cos n + \sum_{n=1}^{44}\sin n = \sqrt{2}\sum_{n=1}^{44}\cos(45 - n) = \sqrt{2}\sum_{n=1}^{44}\cos n$$

$$\implies \sum_{n=1}^{44} \sin n = (\sqrt{2}-1)\sum_{n=1}^{44} \cos n \implies x = \frac{\sum\limits_{n=1}^{44} \cos n}{\sum\limits_{n=1}^{44} \sin n} = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1$$

Thus, $\lfloor 100x \rfloor = \lfloor 100(\sqrt{2}+1) \rfloor = \boxed{241}$.

14. (2002 AIME I Problem 12) Let $F(z) = \frac{z+i}{z-i}$ for all complex numbers $z \neq i$, and let $z_n = F(z_{n-1})$ for all positive integers n. Given that $z_0 = \frac{1}{137} + i$ and $z_{2002} = a + bi$, where a and b are real numbers, find $a + b$.

   [Solution: $a + b = 275$, where $a = 1$ and $b = 274$]

   Iterating F we get:

$$\begin{aligned} F(z) &= \frac{z+i}{z-i} \\ F(F(z)) &= \frac{\frac{z+i}{z-i}+i}{\frac{z+i}{z-i}-i} = \frac{(z+i)+i(z-i)}{(z+i)-i(z-i)} = \frac{z+i+zi+1}{z+i-zi-1} = \frac{(z+1)(i+1)}{(z-1)(1-i)} \\ &= \frac{(z+1)(i+1)^2}{(z-1)(1^2+1^2)} = \frac{(z+1)(2i)}{(z-1)(2)} = \frac{z+1}{z-1}i \\ F(F(F(z))) &= \frac{\frac{z+1}{z-1}i+i}{\frac{z+1}{z-1}i-i} = \frac{\frac{z+1}{z-1}+1}{\frac{z+1}{z-1}-1} = \frac{(z+1)+(z-1)}{(z+1)-(z-1)} = \frac{2z}{2} = z. \end{aligned}$$

   From this, it follows that $z_{k+3} = z_k$, for all k. Thus,

$$z_{2002} = z_{3\cdot 667+1} = z_1 = \frac{z_0+i}{z_0-i} = \frac{(\frac{1}{137}+i)+i}{(\frac{1}{137}+i)-i} = \frac{\frac{1}{137}+2i}{\frac{1}{137}} = 1 + 274i$$

   Thus $a + b = 1 + 274 = \boxed{275}$.

15. (2004 AIME I Problem 13) The polynomial $P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$ has 34 complex roots of the form $z_k = r_k[\cos(2\pi a_k) + i\sin(2\pi a_k)], k = 1, 2, 3, \ldots, 34$, with $0 < a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{34} < 1$ and $r_k > 0$. Given that $a_1 + a_2 + a_3 + a_4 + a_5 = m/n$, where m and n are relatively prime positive integers, find $m + n$.

   [Solution: $m + n = 482$, where $m = 159$ and $n = 323$]

   By using the sum of the geometric series, we see that

$$\begin{aligned} P(x) &= \left(\frac{x^{18}-1}{x-1}\right)^2 - x^{17} = \frac{x^{36}-2x^{18}+1}{x^2-2x+1} - x^{17} \\ &= \frac{x^{36}-x^{19}-x^{17}+1}{(x-1)^2} = \frac{(x^{19}-1)(x^{17}-1)}{(x-1)^2} \end{aligned}$$

   This expression has roots at every 17th root and 19th roots of unity, other than 1. Since 17 and 19 are relatively prime, this means there are no duplicate roots. Thus, a_1, a_2, a_3, a_4 and a_5 are the five smallest fractions of the form $\frac{m}{19}$ or $\frac{n}{17}$ for $m, n > 0$.

   $\frac{3}{17}$ and $\frac{4}{19}$ can both be seen to be larger than any of $\frac{1}{19}, \frac{2}{19}, \frac{3}{19}, \frac{1}{17}, \frac{2}{17}$, so these latter five are the numbers we want to add.

   Thus, $\frac{m}{n} = \frac{1}{19} + \frac{2}{19} + \frac{3}{19} + \frac{1}{17} + \frac{2}{17} = \frac{6}{19} + \frac{3}{17} = \frac{6\cdot 17+3\cdot 19}{17\cdot 19} = \frac{159}{323}$ and so the answer is $m + n = 159 + 323 = \boxed{482}$.

16. (1994 AIME Problem 13) The equation $x^{10}+(13x-1)^{10}=0$ has 10 complex roots $r_1, \overline{r_1}, r_2, \overline{r_2}, r_3, \overline{r_3}, r_4, \overline{r_4}, r_5, \overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}$$

[Solution: 850]

Divide both sides by x^{10} to get

$$1 + \left(13 - \frac{1}{x}\right)^{10} = 0 \implies \left(13 - \frac{1}{x}\right)^{10} = -1 \implies 13 - \frac{1}{x} = \omega$$

where $\omega = \text{cis}(\pi(2n+1)/10)$ and $0 \leq n \leq 9$ is an integer. We see that $\frac{1}{x} = 13 - \omega$. Thus,

$$\frac{1}{x\overline{x}} = (13-\omega)(13-\overline{\omega}) = 169 - 13(\omega + \overline{\omega}) + \omega\overline{\omega} = 170 - 13(\omega + \overline{\omega})$$

Summing over all terms:

$$\frac{1}{r_1\overline{r_1}} + \cdots + \frac{1}{r_5\overline{r_5}} = 5 \cdot 170 - 13(\text{cis}(\pi(1)/10) + \cdots + \text{cis}(\pi(19)/10)) = 850 - 0 = \boxed{850}$$

17. (1998 AIME Problem 13) If $\{a_1, a_2, a_3, \ldots, a_n\}$ is a set of real numbers, indexed so that $a_1 < a_2 < a_3 < \cdots < a_n$, its complex power sum is defined to be $a_1 i + a_2 i^2 + a_3 i^3 + \cdots + a_n i^n$, where $i^2 = -1$. Let S_n be the sum of the complex power sums of all nonempty subsets of $\{1, 2, \ldots, n\}$. Given that $S_8 = -176 - 64i$ and $S_9 = p + qi$, where p and q are integers, find $|p| + |q|$.

[Solution: $|p| + |q| = 368$, where $p = -352$ and $q = 16$]

We note that the number of subsets (for now, including the empty subset, which we will just define to have a power sum of zero) with 9 in it is equal to the number of subsets without a 9. To easily see this, take all possible subsets of $\{1, 2, \ldots, 8\}$. Since the sets are ordered, a 9 must go at the end; hence we can just append a 9 to any of those subsets to get a new one.

Now that we have drawn that bijection, we can calculate the complex power sum recursively. Since appending a 9 to a subset doesn't change anything about that subset's complex power sum besides adding an additional term, we have that $S_9 = 2S_8 + T_9$, where T_9 refers to the sum of all of the $9i^x$.

If a subset of size 1 has a 9, then its power sum must be $9i$, and there is only 1 of these such subsets. There are $\binom{8}{1}$ with $9 \cdot i^2$, $\binom{8}{2}$ with $9 \cdot i^3$, and so forth. So $T_9 = \sum_{k=0}^{8} 9\binom{8}{k} i^{k+1}$. This is exactly the binomial expansion of $9i \cdot (1+i)^8$. We can use De Moivre's Theorem to calculate the power: $(1+i)^8 = (\sqrt{2})^8 \cos(8 \cdot 45) = 16$. Hence $T_9 = 16 \cdot 9i = 144i$, and $S_9 = 2S_8 + 144i = 2(-176 - 64i) + 144i = -352 + 16i$. Thus, $|p| + |q| = |-352| + |16| = \boxed{368}$.

18. (1989 AIME Problem 14) Given a positive integer n, it can be shown that every complex number of the form $r + si$, where r and s are integers, can be uniquely expressed in the base $-n + i$ using the integers $1, 2, \ldots, n^2$ as digits. That is, the equation

$$r + si = a_m(-n+i)^m + a_{m-1}(-n+i)^{m-1} + \cdots + a_1(-n+i) + a_0$$

is true for a unique choice of a non-negative integer m and digits $a_0, a_1, \ldots, a_m$ chosen from the set $\{0, 1, 2, \ldots, n^2\}$, with $a_m \neq 0$. We write

$$r + si = (a_m a_{m-1} \ldots a_1 a_0)_{-n+i}$$

to denote the base $-n+i$ expansion of $r+si$. There are only finitely many integers $k+0i$ that have four-digit expansions

$$k = (a_3a_2a_1a_0)_{-3+i}, \quad a_3 \neq 0$$

Find the sum of all such k.

[Solution: 490]

First, we find the first three powers of $-3+i$:

$(-3+i)^1 = -3+i; (-3+i)^2 = 8-6i; (-3+i)^3 = -18+26i$

So we need to solve the Diophantine equation $a_1 - 6a_2 + 26a_3 = 0 \Longrightarrow a_1 - 6a_2 = -26a_3$.

The minimum the left hand side can go is -54, so $a_3 \leq 2$, so we try cases:

- Case 1: $a_3 = 2$ The only solution to that is $(a_1, a_2, a_3) = (2, 9, 2)$.
- Case 2: $a_3 = 1$ The only solution to that is $(a_1, a_2, a_3) = (4, 5, 1)$.
- Case 3: $a_3 = 0$ a_3 cannot be 0, or else we do not have a four digit number.

So we have the four digit integers $(292a_0)_{-3+i}$ and $(154a_0)_{-3+i}$, and we need to find the sum of all integers k that can be expressed by one of those.

$(292a_0)_{-3+i}$:

We plug the first three digits into base 10 to get $30 + a_0$. The sum of the integers k in that form is 345.

$(154a_0)_{-3+i}$:

We plug the first three digits into base 10 to get $10 + a_0$. The sum of the integers k in that form is 145. The answer is $345 + 145 = \boxed{490}$.