Geometry	Misha Lavrov

Geometry in three dimensions

Western PA ARML Practice — *November 2, 2014*

Warm-up

1. Given a $1 \times 2 \times 4$ brick, what is the shortest distance along the surface of the brick between two opposite vertices?

Problems

1. (AIME 2003) In a regular tetrahedron the centers of the four faces are the vertices of a smaller tetrahedron. What is the ratio of the volumes of the two tetrahedra?
2. (AIME 2005) In a regular octahedron the centers of the six faces are the vertices of a cube. What is the ratio of the volume of the cube to the volume of the octahedron?
3. A $1 \times 1 \times 1$ cube is cut in two parts, with the cross-section in the shape of a regular hexagon. What is the side length of the hexagon?
4. A tetrahedron with side length 1 is cut in two parts, with the cross-section in the shape of a square. What is the side length of the square?
5. A pyramid whose base is a 1×1 square and whose other sides are equilateral triangles is cut in two parts, with the cross-section in the shape of a regular pentagon. What is the side length of the pentagon?
6. (ARML 1993) Two right circular cones have parallel bases (that is, their bases lie in parallel planes), and the apex of each is (at) the center of the base of the other. The cones intersect in circle C. If the areas of the bases are 400 and 900, compute the area of circle C.
7. (ARML 1996) A cylindrical container 10 units high and 4 units in diameter is partially filled with water. The cylinder is tilted so that the water level reaches 9 units up the side of the cylinder at the highest but only 3 units up at the lowest. Compute the volume of water in the cylinder.
8. (AIME 2004) A unicorn is tethered by a 20-foot silver rope to the base of a magician's cylindrical tower whose radius is 8 feet. The rope is attached to the tower at ground level and to the unicorn at a height of 4 feet. The unicorn has pulled the rope taut and the end of the rope is 4 feet from the nearest point on the tower. Find the length of the rope touching the tower.
9. (ARML 2013) A bubble in the shape of a hemisphere of radius 1 is on a tabletop. Inside the bubble are five congruent spherical marbles, four of which are sitting on the table and one which rests atop the others. All marbles are tangent to the bubble, and their centers can be connected to form a pyramid with volume V and with a square base. Compute V.