

Miscellaneous problems

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ARML Practice 1/19/2014

1. Compute

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{99 \cdot 100}.$$

2. Find a closed-form expression for

$$\sum_{k=0}^{n-1} \cos k.$$

Deduce that there is a real number M such that $\sum_{k=0}^{n-1} \cos k < M$ for any n .

(Bonus: find the smallest such M .)

Sums

Solutions

1. Observing that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, we can rewrite the sum as

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right)$$

which simplifies to $1 - \frac{1}{100} = \frac{99}{100}$.

2. Recall the identity $e^{i\theta} = \cos \theta + i \sin \theta$. We have

$$\sum_{k=0}^{n-1} e^{ki} = \frac{e^{ni} - 1}{e^i - 1} = \frac{(e^{ni} - 1)(e^{-i} - 1)}{(e^i - 1)(e^{-i} - 1)} = \frac{e^{(n-1)i} - e^{ni} - e^{-i} + 1}{2 - 2 \cos 1}.$$

The sum $\cos 0 + \cdots + \cos(n-1)$ will be the real part of this expression, equal to

$$\frac{\cos(n-1) - \cos n - \cos 1 + 1}{2 - 2 \cos 1}.$$

Functional equations

1. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that satisfy $f(x) + 2f(1 - x) = x$.
2. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function¹ such that $f(f(n)) = 3n$. Find $f(100)$.

¹i.e., when $x < y$, $f(x) < f(y)$

Functional equations

Solutions

1. Let $y = 1 - x$; then $f(x) + 2f(y) = x$ and $f(y) + 2f(x) = y$. Solving for $f(x)$ and $f(y)$, we get

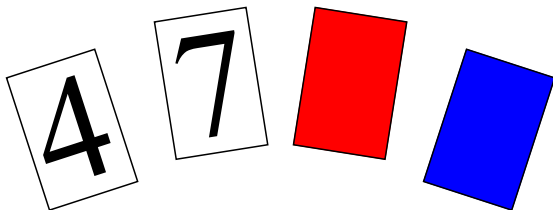
$$f(x) = \frac{2y - x}{3} = \frac{2(1 - x) - x}{3} = \frac{2}{3} - x.$$

2. Casework shows that $f(1) = 2$. From here, we can deduce that $f(2) = f(f(1)) = 3$, $f(3) = f(f(2)) = 6$, $f(6) = f(f(3)) = 9$, $f(9) = f(f(6)) = 18$, and so on.

We will get $f(81) = 162$ and $f(162) = 243$. There are 80 numbers between 81 and 162, and 80 spaces between $f(81)$ and $f(162)$. So in order to satisfy $f(81) < f(82) < f(83) < \dots < f(162)$, we must have $f(x) = x + 81$ on this interval, which means $f(100) = 181$.

Logic and probability

1. Which of the cards below should be flipped to verify the statement “If a card shows an even number on one face, then its opposite face is red?”



2. One in ten thousand pennies are accidentally minted to have Lincoln's face on both sides. A random newly minted penny is chosen, flipped 10 times, and all ten flips come up “heads”. What is the probability that this is one of the error coins?

Logic and probability

Solutions

1. The standard answer is the 4 card (in case the back is blue) and the blue card (in case there is an even number on the back). The way this problem is posed, arguably the 7 card also must be flipped (in case there is an even number on its back, rather than a color).
2. Initially the odds are 1 : 9999 that the penny is an error coin. However, the odds of seeing ten heads in a row are 1024 : 1 in favor of the error coin. Therefore after seeing the coin flips, the odds are 1024 : 9999, and the probability is

$$\frac{1024}{1024 + 9999} = \frac{1024}{11023} \approx 0.093.$$

Questions about dice

1. If two standard six-sided dice are rolled, what is the average value of their sum?
2. Find a way to relabel the sides of a six-sided die with integers such that if two such dice are rolled, the average value remains as above, and the probability of rolling doubles is as high as possible.
3. If two standard six-sided dice are rolled, what is the average value of the larger of the two numbers?

Questions about dice

Solutions

1. One die has average $\frac{1+2+3+4+5+6}{6} = \frac{7}{2}$, so two dice have an average of 7.
2. For the average to remain the same, we want the sum of the six faces to remain 21. This means all six sides cannot be the same. We do the next best thing, and number five out of six sides the same: for example, (1, 4, 4, 4, 4, 4). This yields a probability of $\frac{13}{18}$ of rolling doubles.
3. If X is the value of the larger number, the average of X can be written as

$$\Pr[X \geq 1] + \Pr[X \geq 2] + \cdots + \Pr[X \geq 6].$$

In our case, $\Pr[X < k] = \left(\frac{k-1}{6}\right)^2$: both dice have to come up one of $\{1, \dots, k-1\}$. So $\Pr[X \geq k] = 1 - \frac{(k-1)^2}{36}$, which gives $\frac{161}{36} \approx 4.47$ in the sum above.

The doomsday algorithm

A quick way to figure out the day of the week for any date

The fundamental observation is that in a single year, the dates 4/4, 6/6, 8/8, 10/10, 12/12, and the last day of February all fall on the same day of the week, called the *doomsday* of that year.

The doomsday of 1900 was Wednesday. The doomsday of 2000 was Tuesday.

It helps to think of the days of the week as numbers: Sunday=0, Monday=1, . . . , Saturday=6.

First find the doomsday of your year: it advances by 1 during normal years, and 2 during leap years. So if the last two digits are x , find $x + \lfloor \frac{x}{4} \rfloor \pmod{7}$ and add it to the doomsday of that century. E.g., for 1950, we take $50 + 12 + 3 \equiv 1 + -2 + 3 = 2 = \text{Tuesday}$.

Then count from the nearest date which you know falls on doomsday. E.g. for October 27, 1950, we're 17 days past doomsday, so we're at $2 + 17 \equiv 2 + 3 = 5 = \text{Friday}$.

The doomsday algorithm

Practice

The dates 4/4, 6/6, 8/8, 10/10, 12/12, and the last day of February fall on doomsday.

The doomsday of 1900 was Wednesday (3). The doomsday of 2000 was Tuesday (2). The doomsday of 1700 was Sunday (0).

1. October 31, 1913.
2. July 4, 1776.
3. December 25, 2002.
4. January 31, 2068.
5. April 8, 1888.

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