Number Theory

Theory of Divisors

Misha Lavrov

ARML Practice 9/29/2013

Warm-up

HMMT 2008/2. Find the smallest positive integer n such that $107n$ has the same last two digits as n.

IMO 2002/4. Let n be an integer greater than 1. The positive divisors of n are $d_1, d_2, \ldots, d_k$, where

$$1 = d_1 < d_2 < \cdots < d_k = n.$$

Define $D = d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$.

(a) Prove that $D < n^2$.

(b) Determine all n for which D is a divisor of n^2.

Warm-up

Solutions

1. Two numbers have the same last two digits just when they are the same mod 100, and

$$\begin{aligned} n \equiv 107n \pmod{100} &\Leftrightarrow n \equiv 7n \pmod{100} \\ &\Leftrightarrow 6n \equiv 0 \pmod{100} \\ &\Leftrightarrow 6n = 100k \text{ for some } k \\ &\Leftrightarrow n = 50 \cdot \frac{k}{3}. \end{aligned}$$

   So n must be a multiple of 50, and the smallest such positive number is 50 itself.

2. The IMO problem is left as an exercise.

Divisors of 10000

- We can arrange the divisors of 10000 in a square grid:

1	2	4	8	16
5	10	20	40	80
25	50	100	200	40
125	250	500	1000	2000
625	1250	2500	5000	10000

Divisors of 10000

- We can arrange the divisors of 10000 in a square grid:

1	2	4	8	16
5	10	20	40	80
25	50	100	200	40
125	250	500	1000	2000
625	1250	2500	5000	10000

- Questions:
  - How many divisors of 10000 are divisors of 200?
  - What is the sum of all the divisors of 10000? (Try to figure out how to avoid using brute force.)
  - How many divisors does 10^{100} have?
  - How many divisors does 3600 have?

Competition-level questions

AIME 1998/5. If a random divisor of 10^{99} is chosen, what is the probability that it is a multiple of 10^{88}?

PUMaC 2011/NT A1. The only prime factors of an integer n are 2 and 3. If the sum of the divisors of n (including n itself) is 1815, find n.

Original. How many divisors x of 10^{100} have the property that the number of divisors of x is also a divisor of 10^{100}?

Competition-level questions

Solutions

AIME 1998/5. The divisors of 10^{99} form a 100×100 grid. In the grid, the multiples of 10^{88} are the numbers below and to the right of 10^{88}, which form a 12×12 grid. So the probability is

$$\frac{12 \cdot 12}{100 \cdot 100} = 0.0144.$$

Competition-level questions

Solutions

AIME 1998/5. The divisors of 10^{99} form a 100×100 grid. In the grid, the multiples of 10^{88} are the numbers below and to the right of 10^{88}, which form a 12×12 grid. So the probability is

$$\frac{12 \cdot 12}{100 \cdot 100} = 0.0144.$$

PUMaC 2011/NT A1. First note that 1815 factors as $3 \cdot 5 \cdot 11^2$.

If $n = 2^a \cdot 3^b$, the sum of its divisors is

$$(1 + 2 + 4 + \cdots + 2^a)(1 + 3 + 9 + \cdots + 3^b).$$

The sums of powers of 2 begin $1, 3, 7, 15, 31, \ldots$ and the sums of powers of 3 begin $1, 4, 13, 40, 121, \ldots$. At this point we spot that $15 \cdot 121 = 1815$. This is $1 + 2 + 4 + 8$ times $1 + 3 + 9 + 27 + 81$, so n is $8 \cdot 81 = 648$.

Competition-level questions

Solutions

Original. Since $10^{100} = 2^{100} \cdot 5^{100}$, x must also be of the form $2^a \cdot 5^b$, where $0 \le a \le 100$ and $0 \le b \le 100$.

The divisors of x form their own grid, with $a + 1$ columns (there are $a + 1$ choices for the power of 2, namely $2^0, 2^1, 2^2, \dots, 2^a$) and $b + 1$ rows (there are $b + 1$ choices for the power of 5). The total number of divisors of x is $(a + 1)(b + 1)$.

If this number is also a divisor of 10^{100}, then both $a + 1$ and $b + 1$ must be products of 2's and 5's. There are no further restrictions on x. So $a + 1$ and $b + 1$ can each be one of:

$$1, 2, 4, 8, 16, 32, 64, \quad 5, 10, 20, 40, 80, \quad 25, 50, 100.$$

There are 15 possibilities for a and for b, so there are $15^2 = 225$ possibilities for x.

Taking equations mod n

Pythagorean triples

Problem

If x, y, z are integers and $x^2 + y^2 = z^2$, show that 60 divides xyz.

Taking equations mod n

Pythagorean triples

Problem

If x, y, z are integers and $x^2 + y^2 = z^2$, show that 60 divides xyz.

- All three of x, y, z cannot be odd, since odd + odd = even. So xyz is even.

Taking equations mod n

Pythagorean triples

Problem

If x, y, z are integers and $x^2 + y^2 = z^2$, show that 60 divides xyz.

- All three of x, y, z cannot be odd, since odd + odd = even. So xyz is even.
- Since $1^2 \equiv 2^2 \equiv 1 \pmod 3$, all perfect squares are 0 or 1 mod 3. But $x^2 + y^2 \equiv z^2 \pmod 3$ is not solved by making each of x^2, y^2, and z^2 be 1 mod 3. So one is 0 mod 3, and so xyz is divisible by 3.

Taking equations mod n

Pythagorean triples

Problem

If x, y, z are integers and $x^2 + y^2 = z^2$, show that 60 divides xyz.

- All three of x, y, z cannot be odd, since odd + odd = even. So xyz is even.
- Since $1^2 \equiv 2^2 \equiv 1 \pmod 3$, all perfect squares are 0 or 1 mod 3. But $x^2 + y^2 \equiv z^2 \pmod 3$ is not solved by making each of x^2, y^2, and z^2 be 1 mod 3. So one is 0 mod 3, and so xyz is divisible by 3.
- Mod 5, we have $1^2 \equiv 4^2 \equiv 1$ and $2^2 \equiv 3^2 \equiv -1$. So $x^2 + y^2 \equiv z^2 \pmod 5$ can look like $0 \pm 1 \equiv \pm 1$ or $1 - 1 \equiv 0$. So one of x, y, or z is 0 mod 5, and xyz is divisible by 5.

Taking equations mod n

Pythagorean triples

Problem

If x, y, z are integers and $x^2 + y^2 = z^2$, show that 60 divides xyz.

- All three of x, y, z cannot be odd, since odd + odd = even. So xyz is even.
- Since $1^2 \equiv 2^2 \equiv 1 \pmod 3$, all perfect squares are 0 or 1 mod 3. But $x^2 + y^2 \equiv z^2 \pmod 3$ is not solved by making each of x^2, y^2, and z^2 be 1 mod 3. So one is 0 mod 3, and so xyz is divisible by 3.
- Mod 5, we have $1^2 \equiv 4^2 \equiv 1$ and $2^2 \equiv 3^2 \equiv -1$. So $x^2 + y^2 \equiv z^2 \pmod 5$ can look like $0 \pm 1 \equiv \pm 1$ or $1 - 1 \equiv 0$. So one of x, y, or z is 0 mod 5, and xyz is divisible by 5.
- These mean xyz is divisible by 30. Getting 60 is left as an exercise (Hint: try mod 8.)

Taking equations mod n

Competition-level problems

Original. If x, y, z are integers and $x^2 + y^2 = 3z^2$, show that $x = y = z = 0$.

PUMaC 2007/NT B2. How many positive integers n are there such that $n + 2$ divides $(n + 18)^2$?

British MO 2005/6. Let n be an integer greater than 6. Prove that if $n - 1$ and $n + 1$ are both prime, then $n^2(n^2 + 16)$ is divisible by 720.

PUMaC 2009/NT A3. Find all prime numbers p which can be written as $p = a^4 + b^4 + c^4 - 3$ for some primes (not necessarily distinct) a, b, and c.

Taking equations mod n

Competition-level problems

Original. If x, y, z are integers and $x^2 + y^2 = 3z^2$, show that $x = y = z = 0$. (Hint: mod 3)

PUMaC 2007/NT B2. How many positive integers n are there such that $n+2$ divides $(n+18)^2$? (Hint: mod $n+2$)

British MO 2005/6. Let n be an integer greater than 6. Prove that if $n-1$ and $n+1$ are both prime, then $n^2(n^2+16)$ is divisible by 720. (Hint: mod 2, 3, and 5)

PUMaC 2009/NT A3. Find all prime numbers p which can be written as $p = a^4 + b^4 + c^4 - 3$ for some primes (not necessarily distinct) a, b, and c. (Hint: mod 2, 3, and 5)

Taking equations mod n

Solutions

Original. If $x^2 + y^2 = 3z^2$, then $x^2 + y^2 \equiv 0 \pmod 3$, which is only possible if $x \equiv y \equiv 0 \pmod 3$. So both x and y are divisible by 3, so $x^2 + y^2$ is divisible by 9, and therefore z^2 is divisible by 3.

We now have $(x/3)^2 + (y/3)^2 = 3(z/3)^2$, so the same is true of $x/3, y/3, z/3$. But the numbers cannot have infinitely many factors of 3 unless they are all 0.

Taking equations mod n

Solutions

Original. If $x^2 + y^2 = 3z^2$, then $x^2 + y^2 \equiv 0 \pmod 3$, which is only possible if $x \equiv y \equiv 0 \pmod 3$. So both x and y are divisible by 3, so $x^2 + y^2$ is divisible by 9, and therefore z^2 is divisible by 3.

We now have $(x/3)^2 + (y/3)^2 = 3(z/3)^2$, so the same is true of $x/3, y/3, z/3$. But the numbers cannot have infinitely many factors of 3 unless they are all 0.

PUMaC 2007/NT B2. Since $n + 18 \equiv 16 \pmod{n+2}$, $(n + 18)^2 \equiv 16^2 \pmod{n+2}$ We are given $(n + 18)^2 \equiv 0 \pmod{n+2}$, so $16^2 \equiv 0 \pmod{n+2}$, which means $n + 2$ divides 256. Therefore $n + 2$ is one of $2^2, 2^3, \ldots, 2^8$, which gives 7 solutions.

Taking equations mod n
Solutions

BMO 2005/6. Divisibility by 144 is easy. Neither $n+1$ nor $n-1$ is even, so n must be even; and neither $n+1$ nor $n-1$ is divisible by 3, so n must be divisible by 3. Therefore $n = 6k$, and

$$n^2(n^2+16) = (6k)^2((6k)^2+16) = 144 \cdot k^2(9k^2+4).$$

Now all we need is divisibility by 5. Since neither $n+1$ nor $n-1$ is divisible by 5, we have one of $n \equiv 0, 2, 3 \pmod 5$. Fortunately,

$$\begin{cases} 0^2(0^2+16) = 0 \equiv 0 & \pmod 5 \\ 2^2(2^2+16) = 80 \equiv 0 & \pmod 5 \\ 3^2(3^2+16) = 225 \equiv 0 & \pmod 5. \end{cases}$$

So in all three cases, $n^2(n^2+16)$ is divisible by 5.

Taking equations mod n

Solutions

PUMaC 2009/NT A3. The primes 2, 3, and 5 have the following property: if p is one of 2, 3, or 5, then either $a \equiv 0 \pmod{p}$ or $a^4 \equiv 1 \pmod{p}$. This is easy to check:

$$\begin{cases} 1^4 \equiv 1 & \pmod{2} \\ 1^4 \equiv 2^4 \equiv 1 & \pmod{3} \\ 1^4 \equiv 2^4 \equiv 3^4 \equiv 4^4 \equiv 1 & \pmod{5}. \end{cases}$$

Suppose none of a, b, or c are 2. They are prime, so not divisible by 2. But then

$$p = a^4 + b^4 + c^4 - 3 \equiv 1 + 1 + 1 - 3 \equiv 0 \pmod{2}$$

and p is divisible by 2 (but it's easy to check $p = 2$ doesn't work). So one of a, b, or c has to be 2.

The same argument shows that one of a, b, or c has to be 3, and one has to be 5. This means $p = 2^4 + 3^4 + 5^4 - 3 = 719$.