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SET SYSTEMS WITHOUT A STRONG SIMPLEX*

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Abstract. A *d-simplex* is a collection of $d+1$ sets such that every d of them have nonempty intersection and the intersection of all of them is empty. A *strong d-simplex* is a collection of $d+2$ sets $A, A_1, \ldots, A_{d+1}$ such that $\{A_1, \ldots, A_{d+1}\}$ is a d-simplex, while A contains an element of $\cap_{j \neq i} A_j$ for each i, $1 \le i \le d+1$. Mubayi and Ramadurai [*Combin. Probab. Comput.*, 18 (2009), pp. 441–454] conjectured that if $k \ge d+1 \ge 3$, $n > k(d+1)/d$, and $\mathcal{F}$ is a family of k-element subsets of an n-element set that contains no strong d-simplex, then $|\mathcal{F}| \le \binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a star. We prove their conjecture when $k \ge d+2$ and n is large. The case $k = d+1$ was solved in [M. Feng and X. J. Liu, *Discrete Math.*, 310 (2010), pp. 1645–1647] and [Z. Füredi, *private communication*, St. Paul, MN, 2010]. Our result also yields a new proof of a result of Frankl and Füredi [*J. Combin. Theory Ser.* A, 45 (1987), pp. 226–262] when $k \ge d+2$ and n is large.

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1. Introduction.

For any integer $k \ge 2$, we denote the family of all k-element subsets of $[n] := \{1, \ldots, n\}$ by $\binom{[n]}{k}$. A family $\mathcal{F}$ of sets is a *star* if there exists an element x that lies in all the members in $\mathcal{F}$. We say $\mathcal{F}$ is an *intersecting family* if every two of its members have nonempty intersection. We use $|\mathcal{F}|$ to denote the cardinality of $\mathcal{F}$, i.e., the number of members in $\mathcal{F}$.

The following is one of the most important results in extremal combinatorics.

THEOREM 1.1 (Erdős, Ko, and Rado [5]). *Let $n \ge 2k$ and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family. Then $|\mathcal{F}| \le \binom{n-1}{k-1}$. If $n > 2k$ and equality holds, then $\mathcal{F}$ is a star.*

The forbidden configuration in Theorem 1.1 consists of a pair of disjoint sets. A generalization of this configuration, with geometric motivation, is as follows.

DEFINITION 1.2. *Fix $d \ge 1$. A family of sets is* d-wise-intersecting *if every d of its members have nonempty intersection. A collection of $d+1$ sets $A_1, A_2, \ldots, A_{d+1}$ is a* d-dimensional simplex *(or a* d-simplex*) if it is d-wise-intersecting but not $(d+1)$-wise-intersecting (that is, $\cap_{i=1}^{d+1} A_i = \emptyset$).*

Note that a 1-simplex is a pair of disjoint edges, and Theorem 1.1 states that if $\mathcal{F} \subseteq \binom{[n]}{k}$ with $n \ge 2k$ and $|\mathcal{F}| > \binom{n-1}{k-1}$, then $\mathcal{F}$ contains a 1-simplex. In general, it is conjectured that the same threshold for $\mathcal{F}$ guarantees a d-simplex for every d, $1 \le d \le k-1$. For $d = 2$, this was a question of Erdős [4], while the following general conjecture was formulated by Chvátal.

CONJECTURE 1.3 (Chvátal [2]). *Suppose that $k \ge d+1 \ge 2$ and $n \ge k(d+1)/d$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ contains no d-simplex, then $|\mathcal{F}| \le \binom{n-1}{k-1}$. Equality holds only if $\mathcal{F}$ is a star.*

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Another motivation (see [2, page 358]) is that when we formally let $d = k$, then we obtain the famous open problem of finding the Turán function of the hypergraph $\binom{[k+1]}{k}$, posed by Turán [18] in 1941.

Various partial results on the case $d = 2$ of the conjecture were obtained in [1, 2, 3, 7, 8] until this case was completely settled by Mubayi and Verstraëte [16]. Conjecture 1.3 has been proved by Frankl and Füredi [9] for every fixed k, d if n is sufficiently large. Keevash and Mubayi [13] have also proved the conjecture when k/n and $n/2 - k$ are both bounded away from zero.

Mubayi [14] proved a stability result for the case $d = 2$ of Conjecture 1.3 and conjectured that a similar result holds for larger d.

CONJECTURE 1.4 (Mubayi [14]). *Fix $k \geq d+1 \geq 3$. For every $\delta > 0$, there exist $\epsilon > 0$ and $n_0 = n_0(\epsilon, k)$ such that the following holds for all $n > n_0$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ contains no d-simplex and $|\mathcal{F}| > (1-\epsilon)\binom{n-1}{k-1}$, then there exists a set $S \subseteq [n]$ with $|S| = n-1$ such that $|\mathcal{F} \cap \binom{S}{k}| < \delta\binom{n-1}{k-1}$.*

Subsequently, Mubayi and Ramadurai [15] proved Conjecture 1.4 in a stronger form except in the case $k = d+1$, as follows.

DEFINITION 1.5. *Fix $d \geq 1$. A collection of $d+2$ sets $A, A_1, A_2, \ldots, A_{d+1}$ is a* strong d-simplex *if $\{A_1, A_2, \ldots, A_{d+1}\}$ is a d-simplex and A contains an element of $\cap_{j \neq i} A_j$ for each $i \in [d+1]$.*

Note that a strong 1-simplex is a collection of three sets A, B, C such that $A \cap B$ and $B \cap C$ are nonempty, and $A \cap C$ is empty. Note also that if a family $\mathcal{F}$ contains no d-simplex, then certainly it contains no strong d-simplex (but not vice versa). The main result of Mubayi and Ramadurai [15] can be formulated using asymptotic notation as follows, where $o(1) \to 0$ as $n \to \infty$.

THEOREM 1.6 (Mubayi and Ramadurai [15]). *Fix $k \geq d+2 \geq 3$. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ contain no strong d-simplex. If $|\mathcal{F}| \geq (1-o(1))\binom{n-1}{k-1}$, then there exists an element $x \in [n]$ such that the number of sets of $\mathcal{F}$ omitting x is $o(n^{k-1})$.*

COROLLARY 1.7 (Mubayi and Ramadurai [15]). *Fix $k \geq d+2 \geq 3$. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ contain no strong d-simplex. Then $|\mathcal{F}| \leq (1+o(1))\binom{n-1}{k-1}$ as $n \to \infty$.*

In [13], a similar stability result was proved when k/n and $n/2 - k$ are both bounded away from 0, and the result was used to settle Conjecture 1.3 in this range of n.

Let us describe our contribution. First, we observe that Theorem 1.6 does not hold when $k = d+1$.

PROPOSITION 1.8. *Let $k = d+1 \geq 2$. For every $\epsilon > 0$ there is n_0 such that for all $n \geq n_0$ there is a k-graph $\mathcal{F}$ with n vertices and at least $(1-\epsilon)\binom{n-1}{k-1}$ edges without a strong d-simplex such that every vertex contains at most ϵn^{k-1} edges of $\mathcal{F}$.*

The authors of [15] pointed out that that they were unable to use Theorem 1.6 to prove the corresponding exact result for large n (which would give a new proof of the result of Frankl and Füredi [9]). They subsequently made the following conjecture, which is a strengthening of Chvátal's conjecture.

CONJECTURE 1.9 (Mubayi and Ramadurai [15]). *Let $k \geq d+1 \geq 3$, $n > k(d+1)/d$, and $\mathcal{F} \subseteq \binom{[n]}{k}$ contain no strong d-simplex. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only for a star.*

In section 4, we will prove Conjecture 1.9 for all fixed $k \geq d+2 \geq 3$ and large n.

THEOREM 1.10. *Let $k \geq d+2 \geq 3$ and let n be sufficiently large. If $\mathcal{F} \subseteq \binom{[n]}{k}$ contains no strong d-simplex, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only for a star.*

The case $k = d+1$ behaves somewhat differently from the general case $k \geq d+2$

in that by Proposition 1.8 there are almost extremal configurations very different from a star. In an earlier version of this paper, we were able to prove the case $k = d+1$ of Conjecture 1.9 for all $n \geq 5$ when $k = 3$. Very recently, Feng and Liu [6] solved the case $k = d+1$, using a weight counting method used by Frankl and Füredi in [9]. Independently, Füredi [10] has obtained the same proof, which is short and follows readily from the counting method.

Independently of us, Füredi and Özkahya [12] have re-proved our main result, Theorem 1.10, in a stronger form (for $k \geq d+2$ and large n). Namely, they can additionally guarantee that (in the notation of Definition 1.5) the sets $A_1 \setminus A, \ldots, A_{d+1} \setminus A$ are pairwise disjoint, while the sets $A \setminus A_i, \ldots, A \setminus A_{d+1}$ partition A and have any specified nonzero sizes. Füredi and Özkahya's proof uses a sophisticated version of the delta system method that has been developed in earlier papers such as [9] and [11]. Their method is very different from ours.

The problem of forbidding a d-simplex where we put some extra restrictions on the sizes of certain Boolean combinations of edges has also been studied before, with one particularly interesting paper being that of Csákány and Kahn [3], which uses a homological approach.

Frankl and Füredi's proof [9] of Chvátal's conjecture for a d-simplex for large n is very complicated. Together with the stability result in [15], we obtained a new proof of a stronger result. One key factor seems be that having a special edge A in a strong d-simplex $\{A, A_1, \ldots, A_{d+1}\}$ that contains an element in every d-wise intersection in the d-simplex $\{A_1 \ldots, A_{d+1}\}$ facilitates induction arguments very nicely. This observation, already made in [15], further justifies the interest in strong d-simplices.

2. Some notation and conventions. As is usual in the literature, a collection $\mathcal{F}$ of k-element subsets of a set V is also called a *k-uniform hypergraph* on V, where elements of V are called *vertices* (or *points*) and members of $\mathcal{F}$ are called *hyperedges* (or simply *edges*). We usually identify (hyper)graphs with their edge sets; thus, for example, $|\mathcal{F}|$ denotes the number of edges of $\mathcal{F}$.

Let $\mathcal{F} \subseteq \binom{[n]}{k}$. Recall that a strong d-simplex L in $\mathcal{F}$ consists of $d+2$ hyperedges $A, A_1, A_2, \ldots, A_{d+1}$ such that every d of $A_1, \ldots, A_{d+1}$ have nonempty intersection but $\cap_{i=1}^{d+1} A_i = \emptyset$. Furthermore, A contains an element of $\cap_{j \neq i} A_i$ for each $i \in [d+1]$. This means that we can find some $d+1$ elements $v_1, v_2, \ldots, v_{d+1}$ in A such that for each $i \in [d+1]$, $v_i \in \cap_{j \neq i} A_j$. Note that $v_1, v_2, \ldots, v_{d+1}$ are distinct because no element lies in all of $A_1, \ldots, A_{d+1}$. We call A the *special edge* for L and the set $\{v_1, v_2, \ldots, v_{d+1}\}$ a *special* $(d+1)$*-tuple* for L. (Note that there may be more than one choice of a special $(d+1)$-tuple.)

As usual, the *degree* $d_{\mathcal{F}}(x)$ (or simply $d(x)$) of a vertex x in $\mathcal{F}$ is the number of hyperedges that contain x. For a positive integer p, the *p-shadow* of $\mathcal{F}$ is defined as

$$\Delta_p(\mathcal{F}) = \{S \subseteq [n] : |S| = p, S \subseteq D \text{ for some } D \in \mathcal{F}\}.$$

Also, we let

$$\mathcal{T}_{p+1}(\mathcal{F}) = \{T : T \text{ is a special } (p+1)\text{-tuple for some strong } p\text{-simplex in } \mathcal{F}\}.$$

For each $p \in [k-2]$, let

$$\partial_p^*(\mathcal{F}) = |\Delta_p(\mathcal{F})| + |\mathcal{T}_{p+1}(\mathcal{F})|.$$

Given a vertex x in a hypergraph $\mathcal{F}$, let

$$\mathcal{F} - x = \{D : D \in \mathcal{F}, x \notin D\},$$
$$\mathcal{F}_x = \{D \setminus \{x\} : D \in \mathcal{F}, x \in D\}.$$

3. Proof of Proposition 1.8. We have to show that if $k = d+1$, then there is no stability. Let $\epsilon > 0$ be given. Choose large m such that $\binom{m-1}{k-1} > (1-\epsilon/2)\binom{m}{k-1}$. Let the complete star $\mathcal{H} \subseteq \binom{[m]}{k}$ consist of all k-tuples containing 1. Clearly, $\mathcal{H}$ has no $(k-1)$-simplex. Let $n \to \infty$. A result of Rödl [17] shows that we can find an m-graph $\mathcal{F} \subseteq \binom{[n]}{m}$ with at least $(1-\epsilon/2)\binom{n}{k-1}/\binom{m}{k-1}$ edges such that every two edges of $\mathcal{F}$ intersect in at most $k-2$ vertices. Replace every edge of $\mathcal{F}$ by a copy of the star $\mathcal{H}$. Since no k-subset of $[n]$ is contained in two edges of $\mathcal{F}$, the obtained k-graph $\mathcal{G}$ is well defined.

Next, we observe that $\mathcal{G}$ has no strong $(k-1)$-simplex S. Indeed the special k-set X of S intersects every other edge of S in $k-1$ vertices; thus if X belongs to some copy of the star $\mathcal{H}$, then every other edge of S belongs to the same copy, a contradiction.

The size of $\mathcal{G}$ is at least $(1-\epsilon/2)\binom{n}{k-1}/\binom{m}{k-1} \times (1-\epsilon/2)\binom{m}{k-1} > (1-\epsilon)\binom{n}{k-1}$. Also, by the packing property of $\mathcal{F}$, the number of edges of $\mathcal{G}$ containing any one vertex is at most $\binom{n-1}{k-2}/\binom{m-1}{k-2} \times \binom{m-1}{k-1} < \epsilon n^{k-1}$ when n is large. This establishes Proposition 1.8. $\square$

4. Proof of Theorem 1.10. In order to prove Theorem 1.10, we first establish a general lower bound on $\partial_p^*(\mathcal{F})$ in Theorem 4.5, which is of independent interest. Then we will use Theorem 4.5 to prove Theorem 1.10.

We need several auxiliary lemmas. The first follows readily from Corollary 1.7.

LEMMA 4.1. *For each $k \geq d+2 \geq 3$, there exists an integer $n_{k,d}$ such that for all integers $n \geq n_{k,d}$ if $\mathcal{H} \subseteq \binom{[n]}{k}$ contains no strong d-simplex, then $|\mathcal{H}| \leq 2\binom{n-1}{k-1}$.*

LEMMA 4.2. *For every $k \geq p+2 \geq 3$, there exists a positive constant $\beta_{k,p}$ such that the following holds.*

Let $n_{k,p}$ be defined as in Lemma 4.1. Let $\mathcal{H}$ be a k-uniform hypergraph with $n \geq n_{k,p}$ vertices and $m > 4\binom{n-1}{k-1}$ edges. Then $|\mathcal{T}_{p+1}(\mathcal{H})| \geq \beta_{k,p} m^{\frac{p}{k-1}}$.

Proof. From $m > 4\binom{n-1}{k-1}$, we get $n < \lambda_k m^{\frac{1}{k-1}}$ for some constant λ_k depending only on k. Since $m > 4\binom{n-1}{k-1}$ and $n \geq n_{k,p}$, by Lemma 4.1, $\mathcal{H}$ contains a strong p-simplex L with A being its special edge. Let us remove the edge A from $\mathcal{H}$. As long as $\mathcal{H}$ still has more than $m/2 > 2\binom{n-1}{k-1}$ edges left, we can find another strong p-simplex and remove its special edge from the hypergraph. We can repeat this at least $m/2$ times. This produces at least $m/2$ different special edges. Each special edge contains a special $(p+1)$-tuple. Each special $(p+1)$-tuple is clearly contained in at most $\binom{n-p-1}{k-p-1}$ special edges. So the number of distinct $(p+1)$-tuples in $\mathcal{T}_{p+1}(\mathcal{H})$ is at least $\frac{m}{2\binom{n-p-1}{k-p-1}}$. Using $n < \lambda_k m^{\frac{1}{k-1}}$, we get $|\mathcal{T}_{p+1}(\mathcal{H})| \geq \beta_{k,p} m^{\frac{p}{k-1}}$ for some small positive constant $\beta_{k,p}$ depending on k and p only. $\square$

The next lemma provides a key step to our proof of Theorem 4.5. To some extent, it shows that the notions of strong simplices and special tuples facilitate induction very nicely.

LEMMA 4.3. *Let $k \geq p+2 \geq 3$. Let $\mathcal{F}$ be a k-uniform hypergraph and x a vertex in $\mathcal{F}$. Suppose that $T \in \mathcal{T}_p(\mathcal{F}_x) \cap \Delta_p(\mathcal{F}-x)$. Then $T \cup \{x\} \in \mathcal{T}_{p+1}(\mathcal{F})$.*

Proof. Note that $\mathcal{F}_x$ is $(k-1)$-uniform. By our assumption, T is a special p-tuple for some strong $(p-1)$-simplex $L = \{A, A_1, \ldots, A_p\}$ in $\mathcal{F}_x$, where A is the special edge and $A \supseteq T$. By definition, $\{A_1, \ldots, A_p\}$ is $(p-1)$-wise-intersecting, but $\cap_{i=1}^p A_i = \emptyset$. Suppose that $T = \{v_1, \ldots, v_p\}$, where for each $i \in [p]$ we have $v_i \in \cap_{j \neq i} A_j$. Since $T \in \Delta_p(\mathcal{F}-x)$, there exists $D \in \mathcal{F}-x$ such that $T \subseteq D$.

For each $i \in [p]$, let $A_i' = A_i \cup \{x\}$. Let $A_{p+1}' = D$ and $A' = A \cup \{x\}$. Let $L' = \{A', A_1', \ldots, A_{p+1}'\} \subseteq \mathcal{F}$. We claim that $\{A_1', \ldots, A_{p+1}'\}$ is a p-simplex in $\mathcal{F}$.

Indeed, $x \in \cap_{i=1}^{p} A'_i$. Also, for each $i \in [p]$, $v_i \in \cap_{j\in[p+1]\setminus\{i\}} A'_j$. So, $\{A'_1, \ldots, A'_{p+1}\}$ is p-wise-intersecting. Since $\cap_{i=1}^{p} A_i = \emptyset$, the only element in $\cap_{i=1}^{p} A'_i$ is x. But $x \notin D$ since $D \in \mathcal{F} - x$. So $\cap_{i=1}^{p+1} A'_i = \emptyset$. This shows that $\{A'_1, \ldots, A'_{p+1}\}$ is a p-simplex in $\mathcal{F}$.

Now, let $T' = T \cup \{x\} = \{x, v_1, \ldots, v_p\}$. Then A' contains T'. Let $v_{p+1} = x$. For all $i \in [p+1]$ we have $v_i \in \cap_{j\in[p+1]\setminus\{i\}} A'_j$. Since A' contains $v_1, \ldots, v_{p+1}$, L' is a strong p-simplex in $\mathcal{F}$ with T' being a special $(p+1)$-tuple. That is, $T' \in \mathcal{T}_{p+1}(\mathcal{F})$. ☐

LEMMA 4.4. *Let $k > j \geq 2$. Let $\mathcal{F}$ be a k-graph and let x be a vertex of $\mathcal{F}$. Then*

$$\partial_j^*(\mathcal{F}) \geq \partial_j^*(\mathcal{F} - x) + \partial_{j-1}^*(\mathcal{F}_x).$$

Proof. We want to prove that

$$|\Delta_j(\mathcal{F})| + |\mathcal{T}_{j+1}(\mathcal{F})| \geq |\Delta_j(\mathcal{F} - x)| + |\mathcal{T}_{j+1}(\mathcal{F} - x)| + |\Delta_{j-1}(\mathcal{F}_x)| + |\mathcal{T}_j(\mathcal{F}_x)|. \tag{4.1}$$

Let $T \in \mathcal{T}_j(\mathcal{F}_x)$; that is, T is a special j-tuple in $\mathcal{F}_x$. If $T \in \Delta_j(\mathcal{F} - x)$, we say that T is of Type 1. If $T \notin \Delta_j(\mathcal{F} - x)$, we say T is of Type 2. Suppose $\mathcal{T}_j(\mathcal{F}_x)$ contains a Type 1 special j-tuples and b Type 2 special j-tuples. Then $a + b = |\mathcal{T}_j(\mathcal{F}_x)|$.

For each Type 1 special j-tuple T of $\mathcal{F}_x$, by Lemma 4.3, $T \cup \{x\} \in \mathcal{T}_{j+1}(\mathcal{F})$. Furthermore, it is not in $\mathcal{T}_{j+1}(\mathcal{F} - x)$ since $T \cup \{x\}$ contains x. Hence

$$|\mathcal{T}_{j+1}(\mathcal{F})| \geq |\mathcal{T}_{j+1}(\mathcal{F} - x)| + a. \tag{4.2}$$

For each Type 2 special j-tuple T of $\mathcal{F}_x$, we have $T \in \Delta_j(\mathcal{F})$ since T is contained in some special edge in $\mathcal{F}_x$ which in turn is contained in some edge of $\mathcal{F}$. Also, by our definition of Type 2 special tuples, $T \notin \Delta_j(\mathcal{F} - x)$. Furthermore, T is not of the form $S \cup \{x\}$ since it does not contain x. Also, for each $S \in \Delta_{j-1}(\mathcal{F}_x)$, $S \cup \{x\}$ is an element in $\Delta_j(\mathcal{F})$ that is not in $\Delta_j(\mathcal{F} - x)$. Hence,

$$|\Delta_j(\mathcal{F})| \geq |\Delta_j(\mathcal{F} - x)| + |\Delta_{j-1}(\mathcal{F}_x)| + b. \tag{4.3}$$

When we add (4.2) and (4.3), we obtain the desired inequality (4.1) completing the proof of Lemma 4.4. ☐

THEOREM 4.5. *For all $k \geq p + 2 \geq 3$, there exists a positive constant $c_{k,p}$ such that the following holds: if $\mathcal{F}$ is a k-uniform hypergraph and $m = |\mathcal{F}|$, then $\partial_p^*(\mathcal{F}) \geq c_{k,p} m^{\frac{p}{k-1}}$.*

Proof. Let us remove all isolated vertices from $\mathcal{F}$. Let n denote the number of remaining (i.e., non-isolated) vertices of $\mathcal{F}$. Let $n_{k,p}$ be defined as in Lemma 4.1, which depends only on k and p. Suppose that $n < n_{k,p}$. Since clearly $m \leq \binom{n}{k} < \binom{n_{k,p}}{k}$, $m^{\frac{p}{k-1}}$ is upper bounded by some function of k and p. Hence, $\partial_p^*(\mathcal{F}) \geq \alpha_{k,p} m^{\frac{p}{k-1}}$ for some small enough constant $\alpha_{k,p}$. So, as long as we choose $c_{k,p}$ so that $c_{k,p} \leq \alpha_{k,p}$, the claim holds when $n \leq n_{k,p}$. To prove the general claim, we use induction on p. For each fixed p, we use induction on n noting that when $n \leq n_{k,p}$, the claim has already been verified.

For the basis step, let $p = 1$. Let $c_{k,1} = \min\{\alpha_{k,1}, \beta_{k,1}, 1/4\}$, where $\beta_{k,1}$ is defined in Lemma 4.2. First, suppose that $m \leq 4\binom{n-1}{k-1} < 4n^{k-1}$. Then $n > (m/4)^{\frac{1}{k-1}} > m^{\frac{1}{k-1}}/4$. We have $\partial_1^*(\mathcal{F}) \geq |\Delta_1(\mathcal{F})| = n \geq c_{k,1} m^{\frac{1}{k-1}}$.

Next, suppose that $m > 4\binom{n-1}{k-1}$. By Lemma 4.2, $\partial_1^*(\mathcal{F}) \geq |\mathcal{T}_2(\mathcal{F})| \geq \beta_{k,1} m^{\frac{1}{k-1}} \geq c_{k,1} m^{\frac{1}{k-1}}$. This completes the proof of the basis step.

For the induction step, let $2 \le j \le k-2$. Suppose the claim holds for $p < j$. We prove the claim for $p = j$. We use induction on n. Let

$$c_{k,j} = \min\left\{\alpha_{k,j},\ \beta_{k,j},\ \frac{1}{8k}c_{k-1,j-1}\right\}.$$

Suppose the claim has been verified for k-uniform hypergraphs on fewer than n vertices. Let $\mathcal{F}$ be a k-uniform on n vertices. Suppose $\mathcal{F}$ has m edges. Suppose first that $m > 4\binom{n-1}{k-1}$. By Lemma 4.2, $\partial_j^*(\mathcal{F}) \ge |\mathcal{T}_{j+1}(\mathcal{F})| \ge \beta_{k,j} m^{\frac{j}{k-1}} \ge c_{k,j} m^{\frac{j}{k-1}}$.

Next, suppose that $m \le 4\binom{n-1}{k-1} < 4n^{k-1}$. Then $n > m^{\frac{1}{k-1}}/4$. Hence, the average degree of $\mathcal{F}$ is $km/n < 4km^{\frac{k-2}{k-1}}$. Let x be a vertex in $\mathcal{F}$ of minimum degree d. Then $d < 4km^{\frac{k-2}{k-1}}$.

Note that $\mathcal{F}_x$ is $(k-1)$-uniform with d edges. By the induction hypothesis, we have $\partial^*_{j-1}(\mathcal{F}_x) \ge c_{k-1,j-1} d^{\frac{j-1}{k-2}}$. Also, $\mathcal{F} - x$ is a k-uniform hypergraph on fewer than n vertices (and has $m-d$ edges). By the induction hypothesis, $\partial_j^*(\mathcal{F}-x) \ge c_{k,j}(m-d)^{\frac{j}{k-1}}$. Hence, by Lemma 4.4 we have

$$\partial_j^*(\mathcal{F}) \ge c_{k,j}(m-d)^{\frac{j}{k-1}} + c_{k-1,j-1} d^{\frac{j-1}{k-2}}. \tag{4.4}$$

Recall that $d \le 4km^{\frac{k-2}{k-1}}$. Also, $d \le km/n$. Since we assume that n is large (as a function of k), we may further assume that $d \le m/2$.

Claim 1. *We have* $c_{k,j}(m-d)^{\frac{j}{k-1}} + c_{k-1,j-1}d^{\frac{j-1}{k-2}} \ge c_{k,j} m^{\frac{j}{k-1}}$.

Proof of Claim 1. By the mean value theorem, there exists $y \in (m-d, m) \subseteq (m/2, m)$ such that $c_{k,j}m^{\frac{j}{k-1}} - c_{k,j}(m-d)^{\frac{j}{k-1}} = c_{k,j} d \frac{j}{k-1} y^{\frac{j}{k-1}-1}$. It suffices to prove that $c_{k-1,j-1}d^{\frac{j-1}{k-2}} \ge c_{k,j} d \frac{j}{k-1} y^{\frac{j}{k-1}-1}$, which holds if $c_{k-1,j-1} y^{\frac{k-j-1}{k-1}} \ge c_{k,j} d^{\frac{k-j-1}{k-2}}$. Since $y \ge m/2$, $d \le 4km^{\frac{k-2}{k-1}}$, and $c_{k,j} \le \frac{1}{8k} c_{k-1,j-1}$, one can check that the last inequality holds. ∎

By (4.4) and Claim 1, we have $\partial_j^*(\mathcal{F}) \ge c_{k,j} m^{\frac{j}{k-1}}$. This completes the proof. ∎

Lemma 4.6. *Let $k \ge d+2 \ge 3$. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ contain no strong d-simplex. Let $x \in [n]$. Let $C = \{u_1, \dots, u_d\} \in \Delta_d(\mathcal{F}-x) \cap \Delta_d(\mathcal{F}_x)$. Let $A, B \in \mathcal{F}$ with $x \in A$, $x \notin B$ and $C \subseteq A \cap B$. Let $W \subseteq [n] \setminus (A \cup B)$ such that $|W| = k-d$. For each $i \in [d]$, let $E_W^i = (\{x\} \cup C \cup W) \setminus \{u_i\}$. Then for at least one $i \in [d]$, we have $E_W^i \notin \mathcal{F}$.*

Proof. Suppose on the contrary that, for all $i \in [d]$, $E^i_W \in \mathcal{F}$. Consider the collection $\{A, B, E_W^1, \dots, E_W^d\}$. We have $\cap_{i=1}^d E_W^i = \{x\} \cup W$. For each $i \in [d]$, $\cap_{j\ne i} E_W^j = \{x, u_i\} \cup W$, and so $(\cap_{j \ne i} E_W^j) \cap B = \{u_i\}$. This also implies that $(\cap_{i=1}^d E_W^i) \cap B = \emptyset$. Hence $\{B, E_W^1, \dots, E_W^d\}$ is d-wise-intersecting but not $(d+1)$-wise-intersecting. That is, it is a d-simplex. As A contains an element of each d-wise intersection among $\{B, E_W^1, \dots, E_W^d\}$, these $d+2$ sets form a strong d-simplex in $\mathcal{F}$, a contradiction. ∎

Now, we are ready to prove Theorem 1.10.

Proof of Theorem 1.10. Given d and k, let n be large. Suppose on the contrary that $\mathcal{F} \subseteq \binom{[n]}{k}$ contains no strong d-simplex, $|\mathcal{F}| \ge \binom{n-1}{k-1}$, and $\mathcal{F}$ is not a star. We derive a contradiction. By Theorem 1.6, there exists an element $x \in [n]$ such that $|\mathcal{F} - x| = o(n^{k-1})$ (that is, almost all edges of $\mathcal{F}$ contain x). Let

$$\mathcal{B} = \mathcal{F} - x,$$
$$\mathcal{M} = \left\{ D \in \binom{[n]}{k} : x \in D, D \notin \mathcal{F} \right\}.$$

We call members of $\mathcal{B}$ *bad edges* and members of $\mathcal{M}$ *missing edges*. So, bad edges are those edges in $\mathcal{F}$ not containing x, and missing edges are those k-tuples containing x which are not in $\mathcal{F}$. Since $\binom{n-1}{k-1} \le |\mathcal{F}| = \binom{n-1}{k-1} - |\mathcal{M}| + |\mathcal{B}|$, we have $|\mathcal{B}| \ge |\mathcal{M}|$. Let $b = |\mathcal{B}|$. By the definition of x,

$$b = o(n^{k-1}). \tag{4.5}$$

By Theorem 4.5, $|\Delta_d(\mathcal{B})| + |\mathcal{T}_{d+1}(\mathcal{B})| = \partial_d^*(\mathcal{B}) \ge c_{k,d} b^{\frac{d}{k-1}}$. Since $\mathcal{B} \subseteq \mathcal{F}$, $\mathcal{B}$ contains no strong d-simplex. So, $|\mathcal{T}_{d+1}(\mathcal{B})| = 0$. It follows that

$$|\Delta_d(\mathcal{B})| \ge c_{k,d} b^{\frac{d}{k-1}}.$$

Let

$$S_1 = \Delta_d(\mathcal{B}) \setminus \Delta_d(\mathcal{F}_x) \quad \text{and} \quad S_2 = \Delta_d(\mathcal{B}) \cap \Delta_d(\mathcal{F}_x).$$

We consider two cases.

Case 1. $|S_1| \ge |\Delta_d(\mathcal{B})|/2$.

For any $C \in S_1$ and a set $W \subseteq [n] \setminus (C \cup \{x\})$ of size $k-d-1$, the k-tuple $D = \{x\} \cup C \cup W$ does not belong to $\mathcal{F}$ because $C \notin \Delta_d(\mathcal{F}_x)$. So $D \in \mathcal{M}$. Doing this for each $C \in S_1$ yields a list of $\binom{n-d-1}{k-d-1}|S_1|$ edges (with multiplicity) in $\mathcal{M}$. An edge $D = \{x, y_1, \ldots, y_{k-1}\}$ may appear at most $\binom{k-1}{d}$ times in this list, as it is counted once for each d-subset of $\{y_1, \ldots, y_{k-1}\}$ that appears in S_1. Therefore,

$$b \ge |\mathcal{M}| \ge \frac{\binom{n-d-1}{k-d-1}|S_1|}{\binom{k-1}{d}} \ge c \cdot b^{\frac{d}{k-1}} n^{k-d-1} \tag{4.6}$$

for some properly chosen small positive constant c (depending on k only). Solving (4.6) for b, we get $b \ge c' \cdot n^{k-1}$ for some small positive constant c'. This contradicts (4.5) for sufficiently large n.

Case 2. $|S_2| \ge |\Delta_d(\mathcal{B})|/2$.

By Lemma 4.6, for every d-tuple $C \in S_2$ we may find two edges $A, B \in \mathcal{F}$ such that for every $(k-d)$-set $W \subseteq [n] \setminus (A \cup B)$ there exists $u \in C$ such that $(\{x\} \cup C \cup W) \setminus \{u\} \in \mathcal{M}$. So, we obtain a collection of at least $\binom{n-2k}{k-d}|S_2|$ members of $\mathcal{M}$. Pick an edge $D = \{x, y_1, \ldots, y_{k-1}\}$ in $\mathcal{M}$ and consider its multiplicity in this collection. The edge D may appear each time a $(d-1)$-subset of $\{y_1, \ldots, y_{k-1}\}$ belongs to some d-tuple in S_2. There are $\binom{k-1}{d-1}$ such subsets, and each may be completed to form a d-tuple in at most $n-d+1$ ways by picking the vertex u. Thus,

$$b \ge |\mathcal{M}| \ge \frac{\binom{n-2k}{k-d}|S_2|}{(n-d+1)\binom{k-1}{d-1}} \ge c'' \cdot b^{\frac{d}{k-1}} \cdot n^{k-d-1}$$

for some small positive constant c''. From this, we get $b \ge c''' \cdot n^{k-1}$ for some positive constant c''', which again contradicts (4.5) for sufficiently large n. This completes the proof of Theorem 1.10. $\square$

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