

Mathematics should not be boring

by Walter Noll, March 2003 *

First, let me introduce myself: I have been a Professor of Mathematics at Carnegie Mellon University since 1956. I retired from teaching in 1993, but I am still active with research, writing, and lecturing. I had most of my education in Berlin, Germany, except for one year as a foreign student in France and one year as a foreign graduate student in Bloomington, Indiana, where I received my Ph.D. At various times, I spent a month or more as a visiting professor in Germany, Israel, France, England and Italy. I have a daughter and a son, who both graduated from the Fox Chapel Area High School more than 20 years ago. I was asked to give this lecture as a consequence of a letter to the Pittsburgh Post-Gazette, published in October of 2002. Here is the letter:

“The current debate about the teaching of math is muddled by the failure, on the part of much of the public, to understand that there are two very different kinds of “math”. First, there is arithmetic, taught from grade 1 until about grade 7. It includes addition, subtraction, multiplication, long division, and dealing with fractions and decimals. Learning arithmetic involves a certain amount of pencil and paper drill and rote memorization. Second, there is true mathematics, which starts with geometry and algebra and goes on to trigonometry, calculus and beyond. Here, rote memorization is deadly while conceptual understanding and problem solving ability are essential. My ability in arithmetic is very mediocre. I still have trouble with the multiplication table, and my memory for numbers is atrocious. (I still cannot remember my social security number .) In grade school my arithmetic grades were mostly Cs. Starting with geometry, I was the only one in my class who always got an A. My teacher there realized that I had a special talent for mathematics because I was the only one who could solve tricky problems of geometrical constructions. I have had a very successful career as a professional mathematician.

Arithmetic is the ability to perform calculations quickly and correctly according to fixed recipe. A part of true mathematics is the art of avoiding unnecessary calculation. One of the two or three greatest mathematicians of all history was Karl Friedrich Gauss. When he was 8 years old in a one-room rural grade school, the teacher asked him to add up all the numbers from 1 to 100 to keep him occupied for a while. Instead, Gauss found a way to avoid this stupid calculation and immediately wrote down the answer: 5050. (I challenge the reader to figure out how he did it.) Fortunately, the teacher realized that he was dealing with an unusual mind and saw to it that Gauss obtained the proper education.

In international competitions, American eighth-graders seem to do very well in arithmetic and there is very little wrong with the teaching of arithmetic here. However, when it comes to true mathematics, American twelfth-graders do very

* This is the text of a lecture before the mathematics teachers of the Catholic schools in the Diocese of Pittsburgh, given on April 11, 2003.

poorly. I am not surprised, because true mathematics in high schools in the US is often taught in the same spirit as arithmetic, which is deadly. Unfortunately, there are not enough mathematics majors (as distinct from mathematics education majors) who become high school teachers. For example, almost none of our mathematics graduates at CMU goes into teaching on the high school level.”

When I tell people that I am a mathematician, I very often get a response such as the following: “I am not very good with numbers. I have trouble balancing my checkbook”. Such people confuse being “good with numbers” with having mathematical ability. In fact, there is almost no correlation between the two. I have a colleague at CMU who is extremely “good with numbers” and also a very good mathematician. As I indicated in my letter, I believe I am a very good mathematician but not at all “good with numbers”. Look at two of my report cards: The report card from 5th grade shows a C in arithmetic. The report card from 10th grade shows an A in mathematics and also in physics and chemistry. Note that in the first report card, under the heading “mathematics/arithmetic”, mathematics is crossed out. In German, arithmetic is not considered a part of mathematics. One cannot translate the phrase “If my math is correct” literally into German; you would have to say the equivalent of “If my arithmetic is correct”. This is also true in other languages.

I will now illustrate, with two topics, how mathematics should be presented in high school.

The art of avoiding unnecessary calculation.

To find the sum of all numbers from 1 to 100, call it $S := 1 + 2 + 3 + \dots + 98 + 99 + 100$. Gauss probably considered something like the following arrangement

1	2	3	...	98	99	100
100	99	98	...	3	2	1

The sum in each column is 101 and there are 100 columns. Hence the total sum of all the numbers shown is 100×101 . This is twice S and hence $S = 50 \times 101 = 5,050$. Some of the readers took up the challenge mentioned in my letter, but most had difficulties coming up with a scheme as simple as the one presented above. I suggest that the story of Gauss should be told to the students in their first Algebra class, and then see whether any of them can guess how Gauss did it. Of course, it took a genius to come up with a way, at age 8 in 1785, to avoid such a stupid calculation. At that time, and of course now, mathematicians know a formula for the sum $S := 1+2+3+\dots+(n-2)+(n-1)+n$ of all numbers from 1 up to any number n . This formula can be derived by using an arrangement like the one given above:

1	2	3	...	$n - 2$	$n - 1$	n
n	$n - 1$	$n - 2$...	3	2	1

The sum in each column is $(n + 1)$ and there are n columns. Hence the total of all the numbers shown is $n(n + 1)$. This is twice S and hence:

$$1 + 2 + 3 + \dots + (n - 2) + (n - 1) + n = \frac{n \times (n + 1)}{2} \quad (1)$$

The occasion should be used to tell the students a little bit about Gauss (1777 - 1855) and his achievements. For example, it was Gauss who introduced the "Bell Curve" for statistical distributions. (There is a rather controversial recent book with this title.) He contributed to the study of magnetism, and a unit of magnetic strength, the gauss, is named after him. Electronic instruments often need "degaussing" (demagnetizing). Gauss is considered one of the two or three greatest mathematicians of all time. Even in his lifetime, he was known in Europe as the "Prince of Mathematicians". I often ask people to name a single famous mathematician, living or dead. More often than not I get a blank. Sometimes the name Einstein is mentioned. Einstein was one of the two or three greatest physicists of all time, but his contributions to mathematics were very small. Newton, on the other hand, was both a great physicist and a great mathematician.

When I am at a dinner party and a toast is offered before the meal and people click their glasses, I often ask "how many clicks?", assuming that everyone clicks with everyone else exactly once. I think it is a good idea to pose a question like that to your students. Well, maybe teenagers should not be encouraged to click glasses for a toast. So you could single out a group of perhaps 6 students and have everyone in the group shake hands with everyone else in the group exactly once. You could ask some students not in the group to count the number of handshakes. If no errors are made, the count should be 15. This is an example of unnecessary counting. You can ask your students how one can arrive at this result without actually counting. There are at least two ways:

(1) Line up the students of the group in a row. The first student shakes hands with the second, one handshake. The third student shakes hands with the first and second, two handshakes. The fourth student hands with the previous three, three handshakes. And so forth. The sixth and last student shakes hands with the previous five. The total number of handshakes is

$$1 + 2 + 3 + 4 + 5 = 15 .$$

To compute the sum, one can use the formula (1), with n replaced by 5, giving

$$\frac{5 \times (5 + 1)}{2} = \frac{5 \times 6}{2} = 5 \times 3 = 15$$

(2) Every one of the 6 students shakes hands with the remaining 5. So there seem to be 6 times 5 handshakes. But this method counts each of the handshakes twice, once each from the point of view of one of two people involved. So the total number is one half of 6 times 5:

$$\frac{6 \times 5}{2} = 3 \times 5 = 15 .$$

Of course, the reasoning just described applies to a group of any size. If there are n members in the group then the resulting number of handshakes is

$$\binom{n}{2} := \frac{n(n-1)}{2} \quad (2)$$

The right side is read “ n choose two” and it is actually the formula for the number of different ways two objects can be chosen from a set of n objects. Suppose you have a set having $n + 1$ objects. Separate one object from the rest. There are n ways to choose two objects one of which is the separated object. There are $\binom{n}{2}$ choices of two objects taken from the remaining n objects. Hence using (2) with n replaced by $n + 1$ we obtain

$$\binom{n+1}{2} = n + \binom{n}{2}$$

which can also be verified by using (2) with n replaced by $n + 1$, (2) as it stands, and ordinary algebra.

The formula (2) has many applications. Consider, for example, a round-robin tournament, i.e. a tournament in which every participant, individual or team, plays against every other participant exactly once. If there are n participants, then $\binom{n}{2}$ games must be played. For example, in a chess tournament with an even number $n = 2m$ of participants, there will then be

$$\binom{2m}{2} = \frac{(2m) \times (2m-1)}{2} = m \times (2m-1)$$

games. Suppose you have m tables, so that you could have a round in which m games could be played simultaneously, involving all $2m$ participants. Problem: Can you find an arrangement that makes it possible to have the entire tournament be finished after $(2m-1)$ such rounds? The answer is yes, but it is not at all easy to do. If one of your students can come up with such an arrangement (without looking it up in a book), you may have a future mathematician on your hands.

The number of different ways three objects can be chosen from a set having n members is called “ n choose three” and is given by

$$\binom{n}{3} = \binom{n}{2} \times \frac{n-2}{3} = \frac{n \times (n-1) \times (n-2)}{1 \times 2 \times 3} .$$

You may want to challenge your students to find a reason why this formula is correct. All this could be used as a starting point for a discussion of Pascal's triangle and the binomial theorem.

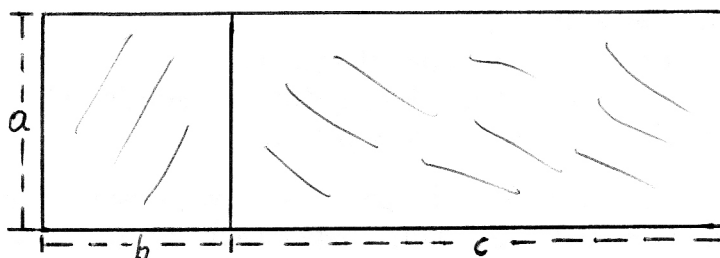
The Theorem of Pythagoras.

When dealing with areas of regions in a plane, it is useful to take the following three principles for granted:

- (P1) If a region is divided into several pieces, then its area is the sum of the areas of the pieces.
- (P2) Congruent regions have the same area.
- (P3) The area of a rectangle is the product of the lengths of two adjacent sides.

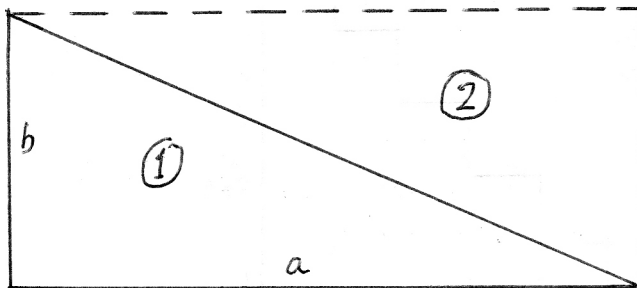
A square is a special case of a rectangle, so, by (P2), its area is the square of its side. Note that areas are measured in square inches, square feet, square meters, etc., when lengths are measured in inches, feet, meters, etc, respectively.

If a rectangle is divided into two rectangles as shown, then (P1) tells us that its area must be the sum of the area of the two pieces.



Using (P3), we find that $a \times (b + c) = a \times b + a \times c$, which is, of course, one of the basic laws of algebra (the distributive law).

We can find the area of a given right triangle (1) putting another right triangle (2) against it as shown.

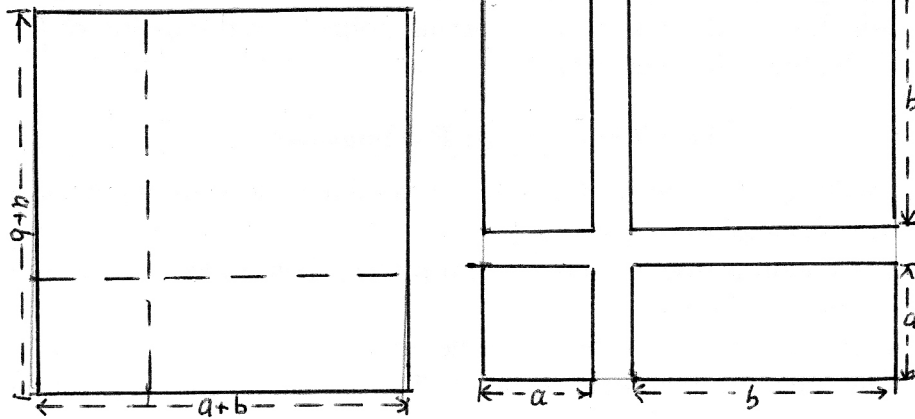


The two triangles make up a rectangle whose area is $a \times b$ by the principle (P3). Since, by (P2), the two triangles have the same area, it follows from (P1) that the area of the given triangle is

$$\text{area} = \frac{1}{2} a \times b \tag{3}$$

We now let positive numbers a and b be given and consider a square with side-length $a + b$. This square can be divided into four pieces as shown, namely

two squares and two rectangles.



Using the principles (P1) and (P2), we find

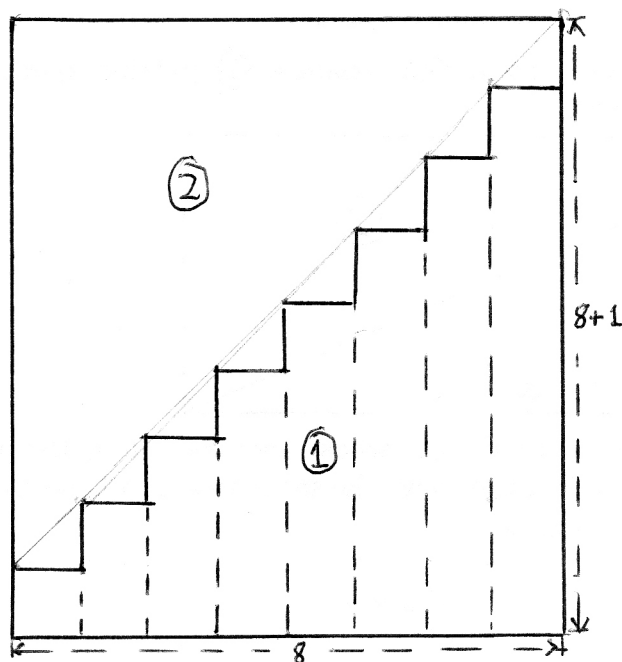
$$(a + b)^2 = a^2 + b^2 + 2ab \quad (4)$$

which is, of course, another basic rule of algebra.

Here are some problems you may assign to your students: Using only the basic principles (P1), (P2), and (P3),

- find a formula for the area of a parallelogram,
- find a formula for the area of an arbitrary (not necessarily right) triangle,
- illustrate the rule $a^2 - b^2 = (a - b) \times (a + b)$.

We now let a natural number n be given, and we consider a rectangle whose sides have lengths n and $n + 1$ units, respectively. By (P3), the area of the rectangle is $n(n + 1)$ square units. The rectangle can be divided into two congruent pieces (1) and (2) as shown for the case when n is 8.



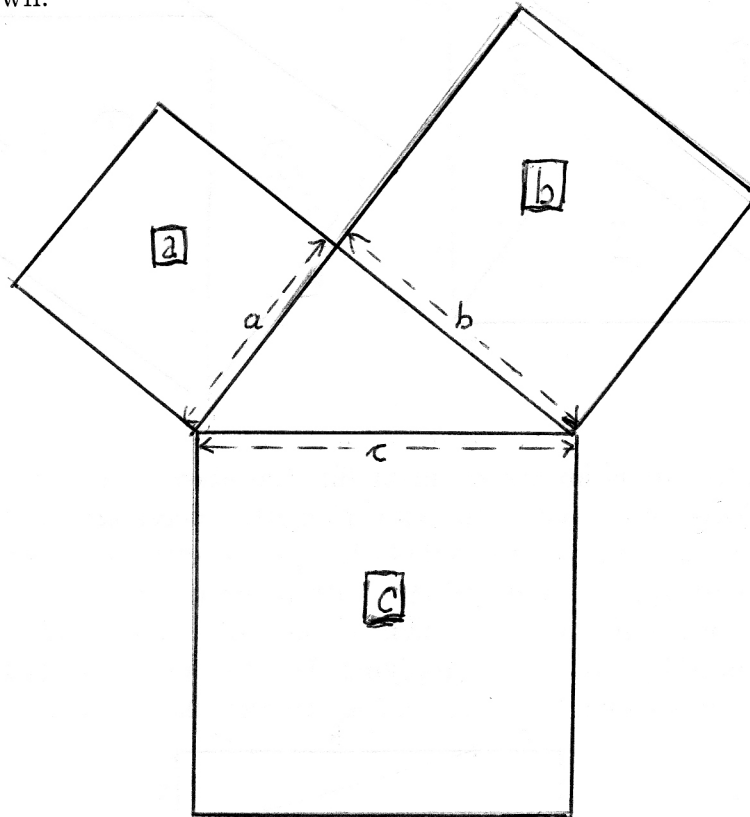
By (P2) the two pieces have the same area and hence, by (P1), the area of

① is $\frac{n(n+1)}{2}$. On the other hand, the piece ① can be divided into n rectangles, which, by (P3) have areas $1, 2, 3, \dots, (n-2), (n-1), n$, respectively. Hence the area of ① is also the sum of all the numbers from 1 to n . We conclude that equation (1) must be valid and have found a geometrical illustration of its proof.

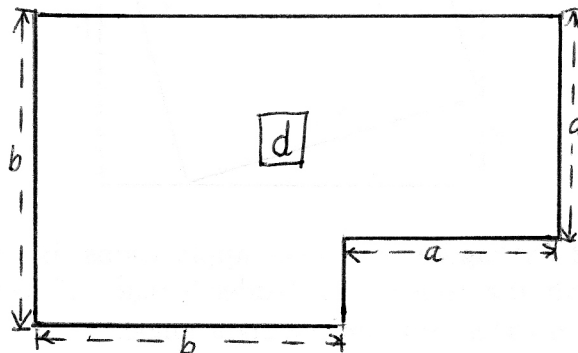
We now consider a right triangle with hypotenuse of length c and legs of lengths a and b . The Theorem of Pythagoras states that

$$c^2 = a^2 + b^2 \quad (5)$$

To prove this theorem we draw squares \boxed{a} , \boxed{b} , and \boxed{c} over the sides of the triangle as shown.



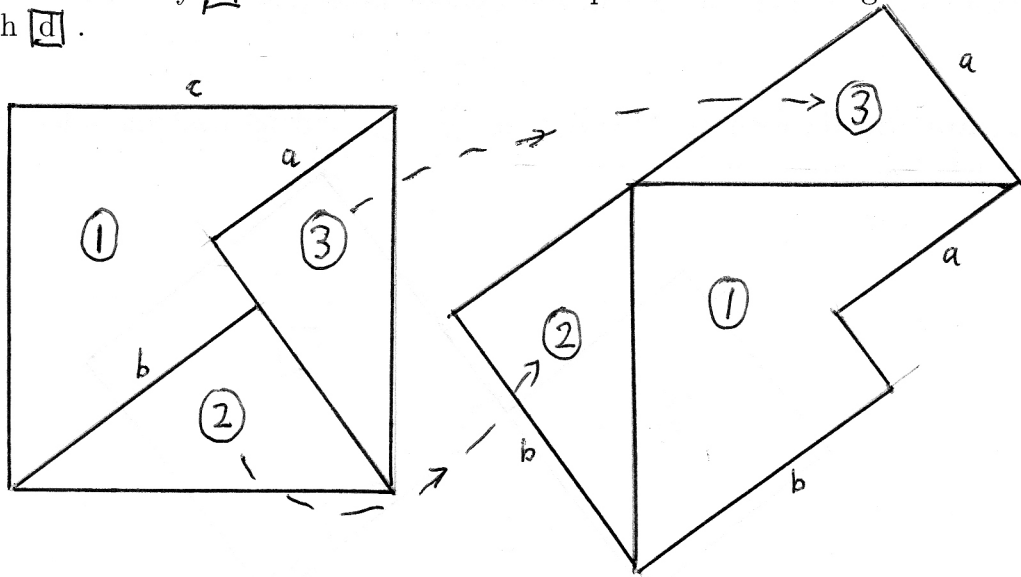
By principle (P3), the theorem can be rephrased as follows: The area of the square \boxed{c} is the sum of the areas of the squares \boxed{a} and \boxed{b} . One can divide the square \boxed{c} into three pieces and then rearrange the three pieces to obtain a region \boxed{d} as shown.



One could challenge the students to figure out a way to do this.

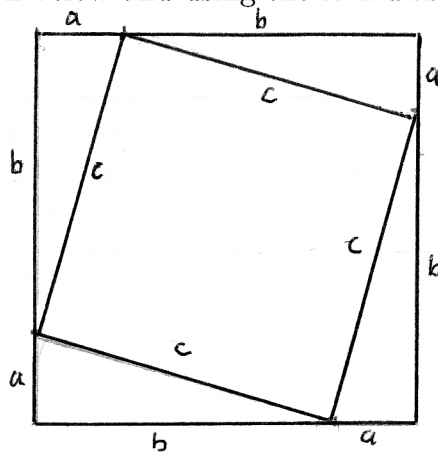
Using the principles (P1) and (P2) in the same way as was done before, it follows that the area of $[d]$ is the same as the area of the square $[c]$. It is clear that the region $[d]$ can be divided into two squares whose areas are a^2 and b^2 , respectively. Hence, using the principle (P1) again, it follows that the assertion (3) of the Theorem of Pythagoras is indeed valid.

Here is the way $[c]$ can be divided and the pieces can be arranged to end up with $[d]$.



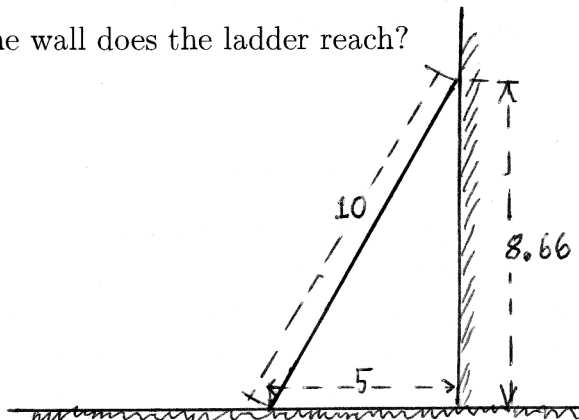
If no student finds an arrangement like the above, one can perhaps give a hint by showing what one of the two triangular pieces are. If students still cannot figure it out, one could tell them how to cut out both triangles 2 and 3 from a cardboard square c and find the rearrangement shown.

Here is a problem you may assign to your students: Find another proof of the Theorem of Pythagoras by applying the three principles (P1), (P2), and (P3) to the picture shown below and using the formulas (3) and (4).



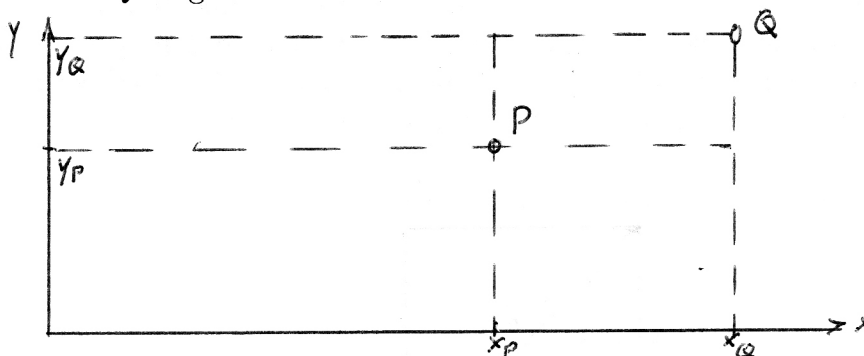
The Theorem of Pythagoras has many applications. Here is a typical practical one: The bottom of a 10 feet long ladder is placed 5 feet from the wall of a house. The wall and the ground form a right angle.

How far up the wall does the ladder reach?



The answer is $\sqrt{10^2 - 5^2} = \sqrt{75}$ feet. Punching this into the square root command of my calculator, I find 8.6600254038 feet. This is not the exact answer, but only an approximation. Actually, we have a case of what I call phony precision. The approximate value 8.66 feet is certainly good enough in practice.

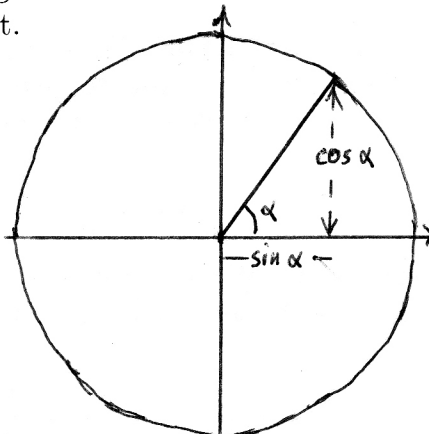
Consider a Cartesian coordinate system for a plane as shown, and let two points P and Q be given.



If the coordinates of P and Q are (x_P, y_P) and (x_Q, y_Q) , respectively, then it follows from the Theorem of Pythagoras that the distance d from P to Q is given by

$$d = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}.$$

The sine and cosine of an angle α are defined as shown in the picture, where the radius of the circle is one unit.



The Theorem of Pythagoras shows that

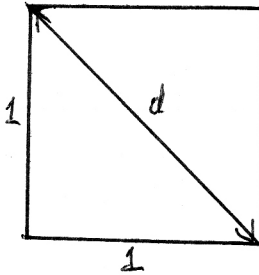
$$(\sin a)^2 + (\cos a)^2 = 1,$$

which is a standard formula of trigonometry.

The converse of the Theorem of Pythagoras is also valid: If the lengths a , b , and c of the sides of a given triangle are such that $c^2 = a^2 + b^2$, then the triangle is a right triangle. This is not self-evident and you should challenge the students to find a proof, i.e., a reason why. Here is the outline of such a proof. It is easy to construct a right triangle whose hypotenuse has length c and one leg has length a . By the Theorem of Pythagoras, the other leg must have length $\sqrt{c^2 - a^2}$, which, by (5), at age 8 in 1785 is b . Hence the lengths of the sides of the constructed right triangle and the lengths of the sides of the given triangle must be the same. Therefore, by one of the theorems about congruence, the two triangles must be congruent and the given triangle must also be right.

The relation (5) is valid if $a = 3$, $b = 4$, and $c = 5$. Therefore, one can produce a right angle by constructing a triangle with sides whose lengths are in the ratio of 3 to 4 to 5. I am told that this fact is used routinely in the construction business now. Perhaps some of the students can find out exactly how. (I can think of several ways.) There is evidence that this method for producing right angles was used in antiquity by the Egyptians when they built the pyramids.

Consider a square whose side length is one unit.



By the Theorem of Pythagoras, the length d of the diagonal is

$$d = \sqrt{1^2 + 1^2} = \sqrt{2}. \quad (6)$$

My calculator gives $\sqrt{2} = 1.414213562$. Again, this is only an approximation. One can ask whether it is possible to give a precise answer in the form of a fraction with numerator n and denominator m :

$$\sqrt{2} = \frac{n}{m}. \quad (7)$$

Suppose that this is possible. We may assume that that the fraction is reduced, i. e. that n and m have no factor in common. It follows from (6) and (7) that

$$2 = d^2 = \left(\frac{n}{m}\right)^2 = \frac{n^2}{m^2}$$

and hence

$$n^2 = 2m^2. \tag{8}$$

It follows that n^2 must be even, which cannot happen unless n itself is even and hence of the form $n = 2k$ for some natural number k . Substituting $2k$ for n in (8), we see that

$$(2k)^2 = 4k^2 = 2m^2 \quad \text{and hence} \quad 2k^2 = m^2.$$

This shows that m^2 and hence m itself must also be even. So, both n and m must be even, which means that they have the factor 2 in common, contradicting the assumption that they have no factor in common.

We conclude that a representation of the form (7) is impossible, and that $\sqrt{2}$ is an irrational number, i.e. a number that is not a ratio of integers. This fact was observed by Greek mathematicians more than 2,000 years ago.

Final observations.

Here are some reasons why I think that, in many cases, the teaching of mathematics on the high-school level is in need of improvement. As one piece of evidence I show you a letter I wrote, in 1980, to the principal of the Fox Chapel Area High School:

“Dear Sir:

I wish to call your attention to some serious deficiencies in the teaching of mathematics by some of the teachers in your school. I learned about these deficiencies because my daughter Virginia (12th grade) and my son Peter (10th grade) attend your school. It seems to me that some of your mathematics teachers do not understand what mathematics is all about. Often, good grades merely reflect speed in doing clerical tasks, good memory, and obedience in following instructions to the letter. True mathematical ability, which has to do with understanding, insight, independent thinking, and ingenuity, is sometimes punished by bad grades rather than rewarded by good ones.

Here are some of the items that have led me to my conclusion:

1. I thought that the purpose of the open house at your school is to have the teachers tell the parents something about the contents and aims of the courses that they are teaching. Most teachers do indeed just that. But Virginia's 11th grade Algebra II teacher spend the entire 10 min. period describing the procedure he used to arrive at his grades, a procedure, incidentally, that struck me as petty and bureaucratic.

2. The teacher mentioned above gave so many problems in his tests that most students could not complete the tests in the allotted time. After this happened he gratuitously berated the students for being “lazy”. Sometimes the subject of the test was not algebra but grade-school arithmetic. Speed in doing arithmetic and routine algebra has nothing to do with mathematical ability. Since I am rather slow in arithmetic, I would probably have flunked these tests myself.

3. On at least one occasion, the same teacher told Virginia that her perfectly good solution to a test problem was wrong. It was not wrong; it was original and hence deviated somewhat from the prescribed procedure.

4. Virginia's teacher on 12th grade Trigonometry gave her 0 points on a test despite the fact that she solved 80% of the problems. As a justification she was told: "You were not taught to do the problems this way". Her procedure in doing these problems was no less sound than the prescribed procedure. This is the worst example of punishment for independent thinking I have ever seen.

5. During the open house for Peter's 10th grade Geometry course the teacher stated: "Let's face it, geometry is mainly a matter of memorizing theorems." Rote-memorizing is antithetical to geometry; it destroys the beauty of the subject. In my entire forty-year career as a student of mathematics, I have never memorized a single thing.

I am sending a copy of this letter to the admissions director of my university to help him properly evaluate the mathematics grade of applicants from your school.

Sincerely yours,
Walter Noll, Ph.D.

Professor of Mathematics, Carnegie Mellon University"

I received a response only from one member of the School Board, none from the school itself.

Most of the mathematicians I know who attended American high schools do not think very much of the instruction they received. Among the few exceptions are those who went the Bronx High School of Science.

My general impression is this: In the US, on the average, elementary schools are quite good, secondary schools need a lot of improvement, colleges range from terrible to excellent, and graduate schools are the best in the world.

The Parade Magazine, which appears as an attachment to many newspapers, including the Pittsburgh Post-Gazette, contains a section called "Ask Marilyn", in which Marilyn Vos Savant, who is alleged to hold the record for the highest IQ, answers questions. Recently, she was asked whether a student should get full credit when, in a geometry test, she solved a problem by a method that was shorter and more elegant than the prescribed method. Marilyn answered no, she should get credit only if she solved the problem by the prescribed method, and then get extra credit for her own method. (Unfortunately, I lost the exact quote.) I was appalled by this answer. The student should be rewarded for finding a better method and not be forced to use the stupid prescribed method. True mathematics is not the skill of carrying out prescribed procedures.

My son is a volunteer tutor to help people who work towards a GED. He recently sent me a copy of a lesson in a book published for this purpose. The title of the lesson is *Working with Right Triangles*. I found it very lacking. The Pythagorean Theorem (called "Relationship" there) is presented without even a hint that it needs a proof. The emphasis is on the wrong issues, and the assignments are repetitive and require too much boring and useless calculation.

The mathematics curriculum in high schools needs some updating. I agree with Steven Pinker, a neuroscientist at MIT, who recently wrote (p.235 of *the blank slate*, Viking, 2002):

“Unfortunately, most curricula have barely changed since medieval times, and are barely changeable, because no one wants to be saying that it is unimportant to learn a foreign language, or English literature, or trigonometry. But no matter how valuable a subject may be, there are only twenty-four hours in a day, and a decision to teach one subject is also a decision not to teach another one. The question is not whether trigonometry is important, but whether it is more important than statistics; ...”

I believe that for most people, statistics is more important than trigonometry. Trigonometry should be taken only by students who plan careers in fields such as engineering, science, architecture, surveying, or carpentry. Even for those students, it may be a good idea to replace trigonometry by vector algebra, which gives a much more efficient way to deal with the issues usually treated with trigonometry. A subject that is important to every citizen is financial mathematics, which needs more emphasis in the curriculum. Every student should learn how to fill in income tax forms. Another subject that should be important to everyone is logic, and some aspects of it could be included in the mathematics curriculum.

You all know about the disastrous results when, in the 1950s, some attempt was made to introduce the “New Math” into secondary and even elementary education. In my research and teaching, I do nothing but “New Math”. However, for various reasons, it has no place in high school, at least not at this time. A pernicious remnant of this “New Math” attempt is the idea that, for example, -2 should be read as “negative two” instead of “minus two”. “Negative” is an adjective, and minus is merely a word for the symbol “-”. The number -2 is the opposite of two, not the “negative” of two. It is correct to say that minus two is negative, but “negative two” is a contradiction in terms, such as “atheist pope”. If one reads the symbol “-” as “negative” one gets such absurdities as “negative a is positive if a is negative” instead of “the opposite of a is positive if a is negative”.

Many people have the mistaken impression that mathematics is a dead subject and nothing new has been discovered over the past hundred years or so. Of course, there have been more discoveries and developments in the past hundred years than in the entire history before. New discoveries in science can, to a large extent, be described in a way that the general public can understand. This is not true for mathematics. When I am invited to give a technical lecture, my non-mathematician friends often asked me to describe what my lecture will be about. I am mostly at a loss to give an answer that is not misleading, because the concepts involved can not be described with ordinary English.