Chapter 5

Geometric Structures.

We assume in this chapter that numbers $r, s \in \tilde{}$, with $r \geq 3$ and $s \in 0..r$, a C^r manifold $\mathcal{M}$ and a C^s linear-space bundle $\mathcal{B}$ over the manifold $\mathcal{M}$ are given. We also assume that both $\mathcal{M}$ and $\mathcal{B}$ have constant dimensions, and put $n := \dim \mathcal{M}$ and $m := \dim \mathcal{B} - \dim \mathcal{M}$. Then we have $n = \dim \mathrm{T}_x\mathcal{M}$ and $m = \dim \mathcal{B}_x$ for all $x \in \mathcal{M}$.

51. Compatible Connections

Let $x \in \mathcal{M}$ be fixed. Let $\mathbf{\Phi}$ be an analytic tensor functor and let $\mathbf{E} \in \mathbf{\Phi}(\mathcal{B}_x)$ be given.

Notation: *We define the mapping*

$$\mathbf{E}^{\diamond} : \mathrm{Tlis}_x\mathcal{B} \to \mathbf{\Phi}(\mathcal{B}) \tag{51.1}$$

by

$$\mathbf{E}^{\diamond}(\mathbf{T}) := \mathbf{\Phi}(\mathbf{T})\mathbf{E} \quad \text{for all} \quad \mathbf{T} \in \mathrm{Tlis}_x\mathcal{B}. \tag{51.2}$$

Since $\mathbf{\Phi}$ is analytic, it is clear that $\mathbf{E}^{\diamond}$ is differentiable at $\mathbf{1}_{\mathcal{B}_x}$.

Proposition 1: *We have $\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^{\diamond} \in \mathrm{Lin}\,(\mathrm{S}_x\mathcal{B}, \mathrm{T}_{\mathbf{E}}\mathbf{\Phi}(\mathcal{B}))$ and, for every bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$,*

$$(\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^{\diamond})\mathbf{s} = \mathbf{A}_{\mathbf{E}}^{\mathbf{\Phi}(\phi)}\mathbf{P}_x\mathbf{s} + \mathbf{I}_{\mathbf{E}}\overset{\bullet}{\mathbf{\Phi}}_x\big(\mathbf{\Lambda}(\mathbf{A}_x^{\phi})\mathbf{s}\big)\mathbf{E} \tag{51.3}$$

for all $\mathbf{s} \in \mathrm{S}_x\mathcal{B}$.

Proof: By using (51.2) and the definition (23.21) of gradient, we obtain the desired result. ∎

Taking the gradient of $\mathbf{E}^{\diamond}\big|_{\mathrm{Lis}\mathcal{B}_x}^{\mathbf{\Phi}(\mathcal{B}_x)}$ at $\mathbf{1}_{\mathcal{B}_x}$, we have

$$\Big(\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^{\diamond}\big|_{\mathrm{Lis}\mathcal{B}_x}^{\mathbf{\Phi}(\mathcal{B}_x)}\Big)\mathbf{L} = \big(\overset{\bullet}{\mathbf{\Phi}}_x(\mathbf{L})\big)\mathbf{E} \tag{51.4}$$

for all $\mathbf{L} \in \mathrm{Lin}\mathcal{B}_x$. For the sake of simplicity, we use the following notation

$$\mathbf{E}^{\circ} := \nabla_{\mathbf{1}_{\mathcal{B}_x}}\Big(\mathbf{E}^{\diamond}\big|_{\mathrm{Lis}\mathcal{B}_x}^{\mathbf{\Phi}(\mathcal{B}_x)}\Big). \tag{51.5}$$

Given $r \in \setminus\{0\}$, we observe from (51.5) that $(r\mathbf{E})^\circ = r\mathbf{E}^\circ$ and hence

$$\mathrm{Null}\,\mathbf{E}^\circ = \mathrm{Null}\,(r\mathbf{E})^\circ. \tag{51.6}$$

It is follows from (51.3) and (51.4) that

$$\mathbf{P}_x = \mathbf{P}_{\mathbf{E}}(\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond) \quad \text{and} \quad (\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond)\mathbf{I}_x = \mathbf{I}_{\mathbf{E}}\mathbf{E}^\circ,$$

i.e. the diagram

$$\begin{array}{ccccc}
\mathrm{Lin}\,\mathcal{B}_x & \xrightarrow{\mathbf{I}_x} & \mathrm{S}_x\mathcal{B} & \xrightarrow{\mathbf{P}_x} & \mathrm{T}_x\mathcal{M} \\
\mathbf{E}^\circ \Big\downarrow & & \nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond \Big\downarrow & & \Big\| \\
\mathbf{\Phi}(\mathcal{B}_x) & \xrightarrow{\mathbf{I}_{\mathbf{E}}} & \mathrm{T}_{\mathbf{E}}\mathbf{\Phi}(\mathcal{B}) & \xrightarrow{\mathbf{P}_{\mathbf{E}}} & \mathrm{T}_x\mathcal{M}
\end{array} \tag{51.7}$$

commutes. And it also clear from (51.3) that

$$\mathbf{A}_{\mathbf{E}}^{\mathbf{\Phi}(\phi)} = (\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond)\mathbf{A}_x^\phi \in \mathrm{Rcon}_{\mathbf{E}}\mathbf{\Phi}(\mathcal{B}) \tag{51.8}$$

for all bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$. More generally, we have

$$(\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond)\mathbf{K} \in \mathrm{Rcon}_{\mathbf{E}}\mathbf{\Phi}(\mathcal{B}) \qquad \text{for all} \quad \mathbf{K} \in \mathrm{Con}_x\mathcal{B}. \tag{51.9}$$

In view of (51.9), the mapping $\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond$ induces the following mapping.

Definition: *We define the mapping*

$$\mathbf{J}_{\mathbf{E}} : \mathrm{Con}_x\mathcal{B} \to \mathrm{Rcon}_{\mathbf{E}}\mathbf{\Phi}(\mathcal{B})$$

by

$$\mathbf{J}_{\mathbf{E}}(\mathbf{K}) := (\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond)\mathbf{K} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Con}_x\mathcal{B}. \tag{51.10}$$

Proposition 2: *The mapping* $\mathbf{J}_{\mathbf{E}}$*, defined in* (51.10)*, is flat. Hence, for every* $\mathbf{D} \in \mathrm{Rng}\,\mathbf{J}_{\mathbf{E}}$*,* $\mathbf{J}_{\mathbf{E}}^{<}(\{\mathbf{D}\})$ *is a flat in* $\mathrm{Con}_x\mathcal{B}$ *with*

$$\dim \mathbf{J}_{\mathbf{E}}^{<}(\{\mathbf{D}\}) = ????.$$

Let a cross section $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$, that is differentiable at $x \in \mathcal{M}$, be given. The gradient of $\mathbf{H}$ at x is a tangent connector of $\mathbf{\Phi}(\mathcal{B})$; i.e. $\nabla_x\mathbf{H} \in \mathrm{Rcon}_{\mathbf{H}(x)}\mathbf{\Phi}(\mathcal{B})$.

Proposition 3: *We have*

$$\nabla_{\mathbf{K}}\mathbf{H} = \mathbf{\Lambda}\big((\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{H}(x)^{\diamond})\mathbf{K}\big)\nabla_x\mathbf{H} \tag{51.11}$$

for all $\mathbf{K} \in \mathrm{Con}_x\mathcal{B}$ *and hence* $\nabla_{\mathbf{K}}\mathbf{H} = \mathbf{0}$ *if and only if* $\mathbf{J}_{\mathbf{H}(x)}(\mathbf{K}) = \nabla_x\mathbf{H}$, *i.e.* $\mathbf{K} \in \mathbf{J}^{<}_{\mathbf{H}(x)}(\{\nabla_x\mathbf{H}\})$.

Proof: The desired result (51.11) follows from (51.8), (41.11), (42.1) and Remark 1 of Sect. 32.

If $\mathbf{K} \in \mathrm{Con}_x\mathcal{B}$ be such that $\nabla_{\mathbf{K}}\mathbf{H} = \mathbf{0}$, then it follows from (51.11) that $\mathbf{\Lambda}\big((\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{H}(x)^{\diamond})\mathbf{K}\big)\nabla_x\mathbf{H} = \mathbf{0}$. Applyiny Prop.1 of Sect.14, we see that this can happen if and only if $(\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{H}(x)^{\diamond})\mathbf{K} = \nabla_x\mathbf{H}$. Since $\mathbf{K} \in \mathrm{Con}_x\mathcal{B}$ was arbitrary, the assertion follows. ∎

Now, let a differentiable cross section $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$ be given.

<u>**Definition**</u>**:** *A connection* $\mathbf{C}\mathcal{M} \to \mathrm{Con}\mathcal{B}$ *is called a* **H-compatible connection** *if* $\nabla_{\mathbf{C}(x)}\mathbf{H} = \mathbf{0}$ *for all* $x \in \mathcal{M}$, *i.e.*

$$\nabla_{\mathbf{C}}\mathbf{H} = \mathbf{0}. \tag{51.12}$$

It clear from Prop.3 that a connection $\mathbf{C}$ is $\mathbf{H}$-compatiable if and only if

$$\mathbf{J}_{\mathbf{H}(x)}(\mathbf{C}(x)) = \nabla_x\mathbf{H} \qquad \text{for all} \quad x \in \mathcal{M}. \tag{51.13}$$

Proposition 4: *Let connectors* $\mathbf{K}_1, \mathbf{K}_2 \in \mathbf{J}^{<}_{\mathbf{H}(x)}(\{\nabla_x\mathbf{H}\})$ *be given and determine* $\mathbf{L} \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\,\mathcal{B}_x)$ *such that* $\mathbf{K}_1 - \mathbf{K}_2 = \mathbf{I}_x\mathbf{L}$*; then we have*

$$\mathbf{H}(x)^{\circ}(\mathbf{L}\mathbf{t}) = \mathbf{0} \qquad \text{for all} \quad \mathbf{t} \in \mathrm{T}_x\mathcal{M}. \tag{51.14}$$

52. Riemannian and Symplectic Bundles

We apply Sect.51 to the case when $\mathbf{\Phi} = \mathrm{Smf}_2$ or Skf_2 (see example (4) of Sect.13).

Let $x \in \mathcal{M}$ be fixed and $\mathbf{E} \in \mathbf{\Phi}(\mathcal{B}_x)$, $\mathbf{\Phi} = \mathrm{Smf}_2$ or Skf_2, be given. We have

$$\mathbf{E}^\circ(\mathbf{M}) = \mathbf{E} \circ (\mathbf{M} \times \mathbf{1}_{\mathcal{B}_x}) + \mathbf{E} \circ (\mathbf{1}_{\mathcal{B}_x} \times \mathbf{M}), \tag{52.1}$$

where $\mathbf{E}^\circ$ is given in (51.5), for every $\mathbf{M} \in \mathrm{Lin}\mathcal{B}_x$.

Proposition 1: *If* $\mathbf{E}$ *is invertiable, then* $\mathbf{E}^\circ$ *is surjective; i.e.*

$$\mathrm{Rng}\,\mathbf{E}^\circ = \mathrm{Sym}_2(\mathcal{B}_x^2,) \quad \text{when} \quad \mathbf{\Phi} = \mathrm{Smf}_2 \tag{52.2}$$

i.e., $\mathbf{E} \in \mathrm{Sym}_2(\mathcal{B}_x^2,)$ *and*

$$\mathrm{Rng}\,\mathbf{E}^\circ = \mathrm{Skw}_2(\mathcal{B}_x^2,) \quad \text{when} \quad \mathbf{\Phi} = \mathrm{Skf}_2 \tag{52.3}$$

i.e., $\mathbf{E} \in \mathrm{Skw}_2(\mathcal{B}_x^2,)$.

Proof: By using (52.1). ∎

Proposition 2: *If* $\mathbf{E}$ *is invertiable, then the flat mapping* $\mathbf{J}_\mathbf{E}$ *defined in* (51.10) *is surjective.*

Proof: The surjectivity follows directly from (51.3), (51.4), (51.5) and the surjectivity of $\mathbf{E}^\circ$. ∎

In view of Prop.2 we see taht, for every $\mathbf{D} \in \mathrm{Rcon}_\mathbf{E}\mathbf{\Phi}(\mathcal{B})$, the preimage $\mathbf{J}_\mathbf{E}^{<}(\{\mathbf{D}\})$ is a flat in $\mathrm{Con}_x\mathcal{B}$. Let $\mathbf{K}_1, \mathbf{K}_2 \in \mathbf{J}_\mathbf{E}^{<}(\{\mathbf{D}\})$ be given and determine $\mathbf{L} \in \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x)$ such that $\mathbf{K}_2 - \mathbf{K}_2 = \mathbf{I}_x\mathbf{L}$. Applying (51.3), we have $\mathbf{0} = \mathbf{J}_\mathbf{E}(\mathbf{K}_2) - \mathbf{J}_\mathbf{E}(\mathbf{K}_1) = \mathbf{E}^\circ(\mathbf{L})$, that is $\mathbf{L} \in \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Null}\,\mathbf{E}^\circ)$. Since $\mathbf{K}_1, \mathbf{K}_2 \in \mathbf{J}_\mathbf{E}^{<}(\{\mathbf{D}\})$ were arbitrary, we conclude that

$$\dim \mathbf{J}_\mathbf{E}^{<}(\{\mathbf{D}\}) = \dim \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Null}\,\mathbf{E}^\circ). \tag{52.4}$$

<u>**Definition:**</u> *A cross section* $\mathbf{G} : \mathcal{M} \to \mathrm{Smf}_2(\mathcal{B})$ *is called a* **Riemannian field** *if, for every* $x \in \mathcal{M}$, $\mathbf{G}(x)$ *is invertiable when regard as element of* $\mathrm{Sym}(\mathcal{B}_x, \mathcal{B}_x^*)$.

A cross section $\mathbf{S} : \mathcal{M} \to \mathrm{Skf}_2(\mathcal{B})$ *is called a* **symplectic field of** $\mathcal{B}$ *if, for every* $x \in \mathcal{M}$, $\mathbf{S}(x)$ *is invertiable when regard as element of* $\mathrm{Skw}(\mathcal{B}_x, \mathcal{B}_x^*)$.

We say that $\mathcal{B}$ *is a* C^s **Riemannian linear space bundle** *if it is endowed with additional structure by the prescription of a* C^s *Riemannian field.*

We say that $\mathcal{B}$ *is a* C^s **symplectic linear space bundle** *if it is endowed with additional structure by the prescription of a* C^s *symplectic field.*

Remark 1: A symplectic field of $\mathcal{B}$ exist if and only if, for every $x \in \mathcal{M}$, $m := \dim \mathcal{B}_x$ is even (see Sect.11). If m is odd, then

$$\mathrm{Skw}(\mathcal{B}_x, \mathcal{B}_x^*) \cap \mathrm{Lis}(\mathcal{B}_x, \mathcal{B}_x^*) = \emptyset.$$

∎

Proposition 3: *If* $\mathbf{G} : \mathcal{M} \to \mathrm{Smf}_2(\mathcal{B})$ *is a Riemannian field, then*

$$\dim \mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x \mathbf{G}\}) = n\binom{m}{2} \quad \text{for all} \quad x \in \mathcal{M}. \tag{52.5}$$

If $\mathbf{S} : \mathcal{M} \to \mathrm{Skf}_2(\mathcal{B})$ *is a symplectic field, then*

$$\dim \mathbf{J}^{<}_{\mathbf{S}(x)}(\{\nabla_x \mathbf{S}\}) = n\binom{m+1}{2} \quad \text{for all} \quad x \in \mathcal{M}. \tag{52.6}$$

Proof: It following easily from (52.4), (52.2) and (52.3). ∎

Remark 2: Let $\mathbf{G}$ be a Riemannian field and $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ be a $\mathbf{G}$-compatible connection. Let $\mathbf{L} : \mathcal{M} \to \mathrm{Lis}\,\mathcal{B}$ be a cross section with $\nabla_{\mathbf{C}}\mathbf{L} = \mathbf{0}$ be given. Then, it follows from $\nabla_{\mathbf{C}}\mathbf{G} = \mathbf{0}$ and $\nabla_{\mathbf{C}}\mathbf{L} = \mathbf{0}$ that $\nabla_{\mathbf{C}}(\mathbf{G} \circ (\mathbf{L} \times \mathbf{L})) = \mathbf{0}$. Hence, the Riemannian field $\mathbf{H} := \mathbf{G} \circ (\mathbf{L} \times \mathbf{L})$ satisfies $\nabla_{\mathbf{C}}\mathbf{H} = \mathbf{0}$. ∎

53. Riemannian and Symplectic Manifolds.

Definition: *We say that* $\mathcal{M}$ *is a* **Riemannian manifold** *if the tangent bundle* $\mathrm{T}\mathcal{M}$ *is endowed with additional structure by the prescription of a* C^{r-1} *Riemannian field.*

We say that $\mathcal{M}$ *is a* **symplectic manifold** *if the tangent bundle* $\mathrm{T}\mathcal{M}$ *is endowed with additional structure by the prescription of a* C^{r-1} *symplectic field.*

Let a Riemannian field $\mathbf{G} : \mathcal{M} \to \mathrm{Sym}^{\mathrm{inv}}(\mathrm{T}\mathcal{M}, \mathrm{T}\mathcal{M}^*)$ of class C^{r-1} be given.

Proposition 1: *For every* $x \in \mathcal{M}$*, the restriction*

$$\mathbf{T}_x\big|_{\mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x \mathbf{G}\})} : \mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x \mathbf{G}\}) \to \mathrm{Skw}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}) \tag{53.1}$$

of the torsion mapping $\mathbf{T}_x$ *is bijective.*

Proof: Given $x \in \mathcal{M}$. If $\mathbf{K}_1, \mathbf{K}_2 \in \mathrm{Con}_x(\mathrm{T}\mathcal{M}, \mathcal{M})$, then we have $\mathbf{T}_x(\mathbf{K}_1) = \mathbf{T}_x(\mathbf{K}_2)$ if and only if $\mathbf{K}_1 - \mathbf{K}_2 = \mathbf{I}_x\mathbf{L}$ for some $\mathbf{L} \in \mathrm{Sym}_2((\mathrm{T}_x\mathcal{M})^2, \mathrm{T}_x\mathcal{M})$ and hence

$$(\mathbf{G}(x)\mathbf{L})(\mathbf{t}, \mathbf{b}, \mathbf{d}) = (\mathbf{G}(x)\mathbf{L})(\mathbf{b}, \mathbf{t}, \mathbf{d}) \tag{53.2}$$

for all $\mathbf{t}, \mathbf{b}, \mathbf{d} \in \mathrm{T}_x\mathcal{M}$.

Let $\mathbf{K}_1, \mathbf{K}_2 \in \mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x \mathbf{G}\})$ with $\mathrm{T}_x(\mathbf{K}_1) = \mathrm{T}_x(\mathbf{K}_2)$ be given and determining $\mathbf{L} \in \mathrm{Lin}_2((\mathrm{T}_x\mathcal{M})^2, \mathrm{T}_x\mathcal{M})$ such that $\mathbf{K}_1 - \mathbf{K}_2 = \mathbf{I}_x\mathbf{L}$. Applying (52.1), (51.14) and (53.2), we have

$$\begin{aligned}(\mathbf{G}(x)\mathbf{L})(\mathbf{t}, \mathbf{b}, \mathbf{d}) &= -(\mathbf{G}(x)\mathbf{L})(\mathbf{t}, \mathbf{d}, \mathbf{b}) = -(\mathbf{G}(x)\mathbf{L})(\mathbf{d}, \mathbf{t}, \mathbf{b}) = \\ &= (\mathbf{G}(x)\mathbf{L})(\mathbf{d}, \mathbf{b}, \mathbf{t}) = (\mathbf{G}(x)\mathbf{L})(\mathbf{b}, \mathbf{d}, \mathbf{t}) = \\ &= -(\mathbf{G}(x)\mathbf{L})(\mathbf{b}, \mathbf{t}, \mathbf{d}) = -(\mathbf{G}(x)\mathbf{L})(\mathbf{t}, \mathbf{b}, \mathbf{d})\end{aligned}$$

for all $\mathbf{t}, \mathbf{b}, \mathbf{d} \in \mathrm{T}_x\mathcal{M}$. This shown that $\mathbf{G}(x)\mathbf{L} = \mathbf{0}$. Since $\mathbf{G}(x)$ is invertible, we observe that $\mathbf{L} = \mathbf{0}$ and hence $\mathbf{K}_1 = \mathbf{K}_2$. In other words, the restriction

$$\mathbf{T}_x\big|_{\mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x \mathbf{G}\})} : \mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x \mathbf{G}\}) \to \mathrm{Skw}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}) \tag{53.3}$$

of the flat mapping $\mathbf{T}_x$ is injective and hence bijective. Since $x \in \mathcal{M}$ was arbitrary, the assertion follows. ∎

Proposition 2: *For every $x \in \mathcal{M}$, we have*

$$\mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x \mathbf{G}\}) = \Big\{ \mathbf{K} - \frac{1}{2}\mathbf{I}_x \mathbf{G}(x)^{-1}\big(\mathrm{S}\,(\nabla_{\mathbf{K}}\mathbf{G})\big) \Big| \, \mathbf{K} \in \mathrm{Con}_x(\mathcal{TM}, \mathcal{M}) \Big\} \tag{53.4}$$

where

$$\big(\mathrm{S}\,(\nabla_{\mathbf{K}}\mathbf{G})\big) = \nabla_{\mathbf{K}}\mathbf{G} + \nabla_{\mathbf{K}}\mathbf{G}^{\sim(1,2)} - \nabla_{\mathbf{K}}\mathbf{G}^{\sim(1,3)}.$$

Moreover, if $\mathbf{K}_1, \mathbf{K}_2 \in \mathrm{Con}_x(\mathcal{TM}, \mathcal{M})$ *with* $\mathbf{T}_x(\mathbf{K}_1) = \mathbf{T}_x(\mathbf{K}_2)$, *i.e.*

$$\mathbf{K}_1 - \mathbf{K}_2 \in \{\mathbf{I}_x\}\mathrm{Sym}_2(\mathcal{T}_x\mathcal{M}^2, \mathcal{T}_x\mathcal{M})),$$

then we have

$$\begin{aligned} &\mathbf{K}_1 - \frac{1}{2}\mathbf{I}_x \mathbf{G}(x)^{-1}\big(\nabla_{\mathbf{K}_1}\mathbf{G} + \nabla_{\mathbf{K}_1}\mathbf{G}^{\sim(1,2)} - \nabla_{\mathbf{K}_1}\mathbf{G}^{\sim(1,3)}\big) \\ &\qquad = \mathbf{K}_2 - \frac{1}{2}\mathbf{I}_x \mathbf{G}(x)^{-1}\big(\nabla_{\mathbf{K}_2}\mathbf{G} + \nabla_{\mathbf{K}_2}\mathbf{G}^{\sim(1,2)} - \nabla_{\mathbf{K}_2}\mathbf{G}^{\sim(1,3)}\big). \end{aligned} \tag{53.5}$$

Proof: By (41.8), we have

$$\begin{aligned} \big((\Box_x\mathbf{G})\mathbf{I}_x\mathbf{G}(x)^{-1}\nabla_{\mathbf{K}}\mathbf{G}\big)(\mathbf{s},\mathbf{t},\mathbf{u}) &= \nabla_{\mathbf{K}}\mathbf{G}(\mathbf{s},\mathbf{t},\mathbf{u}) + \nabla_{\mathbf{K}}\mathbf{G}(\mathbf{s},\mathbf{u},\mathbf{t}), \\ \big((\Box_x\mathbf{G})\mathbf{I}_x\mathbf{G}(x)^{-1}\nabla_{\mathbf{K}}\mathbf{G}^{\sim(1,2)}\big)(\mathbf{s},\mathbf{t},\mathbf{u}) &= \nabla_{\mathbf{K}}\mathbf{G}(\mathbf{t},\mathbf{s},\mathbf{u}) + \nabla_{\mathbf{K}}\mathbf{G}(\mathbf{u},\mathbf{s},\mathbf{t}), \\ \big((\Box_x\mathbf{G})\mathbf{I}_x\mathbf{G}(x)^{-1}\nabla_{\mathbf{K}}\mathbf{G}^{\sim(1,3)}\big)(\mathbf{s},\mathbf{t},\mathbf{u}) &= \nabla_{\mathbf{K}}\mathbf{G}(\mathbf{t},\mathbf{u},\mathbf{s}) + \nabla_{\mathbf{K}}\mathbf{G}(\mathbf{u},\mathbf{t},\mathbf{s}); \end{aligned} \tag{53.6}$$

for all $\mathbf{s},\mathbf{t},\mathbf{u} \in \mathcal{T}_x\mathcal{M}$. Observing $\nabla_{\mathbf{K}}\mathbf{G} \in \mathrm{Lin}\big(\mathcal{T}_x\mathcal{M}, \mathrm{Sym}_2(\mathcal{T}_x\mathcal{M}^2,)\big)$, we see that (53.4)) follows easily from (53.6). ∎

The more general version of "the fundamental theorem of Riemannian geometry" follows immediately from Prop. 1:

Fundamental Theorem of Riemannian Geometry (with torsion):
For every prescribed torsion field $\mathbf{L} : \mathcal{M} \to \mathrm{Skw}_2(\mathrm{T}\mathcal{M}^2, \mathrm{T}\mathcal{M})$ *of class* C^s, $s \in 0..r-2$, *there is exactly one* $\mathbf{G}$*-compatible connection* $\mathbf{C}$, *i.e. one satisfying* $\nabla_{\mathbf{C}}\mathbf{G} = \mathbf{0}$, *such that* $\mathbf{T}(\mathbf{C}) = \mathbf{L}$. $\mathbf{C}$ *is of class* C^s.

Remark 1: When $\mathbf{L} = \mathbf{0}$, the corresponding connection is called the **Levi-Cività connection**. ∎

Remark 2: It follows from Theorem 3 that for every connection $\mathbf{C}' : \mathcal{M} \to \mathrm{Con}\,\mathcal{TM}$ of class C^s, $s \in 0..r-2$, there is exactly one connection $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\,\mathcal{TM}$ such that $\mathbf{T}(\mathbf{C}) = \mathbf{T}(\mathbf{C}')$ and $\nabla_{\mathbf{C}}\mathbf{G} = \mathbf{0}$. Moreover, in view of Prop. 2, we have

$$\mathbf{C} = \mathbf{C}' - \frac{1}{2}\mathbf{I}\mathbf{G}^{-1}\big(\nabla_{\mathbf{C}'}\mathbf{G} - \nabla_{\mathbf{C}'}\mathbf{G}^{\sim(1,2)} + \nabla_{\mathbf{C}'}\mathbf{G}^{\sim(1,3)}\big). \tag{53.7}$$

Now let a connection $\mathbf{C} : \to \mathrm{Con}\mathrm{T}\mathcal{M}$ be given. We may define, for each $x \in \mathcal{M}$, a mapping

$$\mathbf{A}_x^{\mathbf{C}} : \mathrm{Con}_x\mathrm{T}\mathcal{M} \to \mathrm{Sym}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}) \tag{53.8}$$

by

$$\mathbf{A}_x^{\mathbf{C}}(\mathbf{K}) := \mathbf{\Lambda}(\mathbf{C}(x))\mathbf{K} + \big(\mathbf{\Lambda}(\mathbf{C}(x))\mathbf{K}\big)^{\sim} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Con}_x\mathrm{T}\mathcal{M}. \tag{53.9}$$

Let a symplectic field $\mathbf{S} : \mathcal{M} \to \mathrm{Skw}^{\mathrm{inv}}(\mathrm{T}\mathcal{M}, \mathrm{T}^*\mathcal{M})$ of class C^{r-1} be given.

Proposition 3: *For every $x \in \mathcal{M}$, the restriction*

$$\mathbf{A}_x^{\mathbf{C}}\big|_{\mathbf{J}^{<}_{\mathbf{S}(x)}(\{\nabla_x\mathbf{S}\})} : \mathbf{J}^{<}_{\mathbf{S}(x)}(\{\nabla_x\mathbf{S}\}) \to \mathrm{Sym}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}) \tag{53.10}$$

of the mapping $\mathbf{A}_x^{\mathbf{C}}$ is bijective.

Proof: Similar to the proof of Prop. 1. ∎

Proposition 4: *For every connection $\mathbf{C}$ and each prescribed symmetric field $\mathbf{L} : \mathcal{M} \to \mathrm{Sym}_2(\mathrm{T}\mathcal{M}^2, \mathrm{T}\mathcal{M})$ of class C^s, $s \in 0..r-2$, there is exactly one $\mathbf{S}$-compatible connection $\mathbf{K}$, i.e. one satisfying $\nabla_{\mathbf{K}}\mathbf{S} = \mathbf{0}$, such that $\mathbf{A}^{\mathbf{C}}(\mathbf{K}) = \mathbf{L}$. $\mathbf{K}$ is of class C^s.*

Proof: It follows immediately from Prop.3. ∎

Notes 53

(1) The proof of the Fundamental Theorem of Riemannian Geometry given here is modelled on the proof given by Noll in [N1].

(2) In [Sp], Spivak, M. stated: "Perhaps its only defect [of the fundamental theorem of Riemannian geometry] is the restriction to symmetric connections." We show that this restriction is not needed.

54. Identities

Let a C^r, $r \geq 2$, Riemannian manifold $\mathcal{M}$ with the Riemannian-field $\mathbf{G}$ be given. Assume that $\dim \mathcal{M} \geq 2$.

For every $A, B \in \mathfrak{X}(\mathrm{T}\mathcal{M})$ and a connection $\mathbf{C} : \mathcal{M} \to \mathrm{Con}(\mathrm{T}\mathcal{M})$, we use the following notations

$$\langle A, B\rangle := \mathbf{G}(A, B) \quad \text{and} \quad \nabla_A B := (\nabla_{\mathbf{C}} B)A.$$

Proposition 1: *A connection* $\mathbf{C}$ *on a Riemannian manifold* $\mathcal{M}$ *is compatible with the Riemannian-field* $\mathbf{G}$ *if and only if*

$$A\langle B, D\rangle = \langle \nabla_A B, D\rangle + \langle B, \nabla_A D\rangle \tag{54.1}$$

for all $A, B, D \in \mathfrak{X}(\mathrm{T}\mathcal{M})$.

Proof: Taking the covariant gradient of $\mathbf{G} \circ (B, D)$ with respect to $\mathbf{C}$, we obtain

$$(\nabla_{\mathbf{C}}(\mathbf{G} \circ (B, D)))A = \mathbf{G}((\nabla_{\mathbf{C}} B)A, D) + \mathbf{G}(B, (\nabla_{\mathbf{C}} D)A). \\ + (\nabla_{\mathbf{C}} \mathbf{G})(A, B, D)$$

The equation (I.1) holds if and only if $\nabla_{\mathbf{C}} \mathbf{G} = \mathbf{0}$. ∎

For the sake of simplification, we adapt the following notation

$$\langle\langle X, Y, Z, T\rangle\rangle := \langle \mathbf{R}(X, Y)Z, T\rangle \quad \text{for all} \quad X, Y, Z, T \in \mathfrak{X}(\mathrm{T}\mathcal{M}),$$

where $\mathbf{R} := \mathbf{R}(\mathbf{C})$ is the curvature field for a given connection $\mathbf{C}$. Also recall that

$$\mathbf{R}(X, Y, Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

for all $X, Y, Z \in \mathfrak{X}(\mathrm{T}\mathcal{M})$.

Proposition 2: *Let* $\mathbf{C}$ *be a connection on a Riemannian manifold* $\mathcal{M}$ *which is compatible with the Riemannian-field* $\mathbf{G}$*, then we have*

$$\langle\langle X, Y, Z, T\rangle\rangle = -\langle\langle X, Y, T, Z\rangle\rangle \tag{54.2}$$

for all $X, Y, Z, T \in \mathfrak{X}(\mathrm{T}\mathcal{M})$.

Proof: To prove (I.2) is equivalent to show

$$0 = \langle\langle X, Y, Z, Z\rangle\rangle = \langle \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, Z\rangle.$$

Applying (I.1), we have

$$\langle \nabla_Y \nabla_X Z, Z\rangle = Y\langle \nabla_X Z, Z\rangle - \langle \nabla_X Z, \nabla_Y Z\rangle$$

and

$$\langle \nabla_X \nabla_Y Z, Z\rangle = X\langle \nabla_Y Z, Z\rangle - \langle \nabla_Y Z, \nabla_X Z\rangle.$$

Hence

$$\langle\!\langle X, Y, Z, Z\rangle\!\rangle = Y\langle \nabla_X Z, Z\rangle - X\langle \nabla_Y Z, Z\rangle + \langle \nabla_{[X,Y]} Z, Z\rangle.$$

It follows from (I.1) and the symmetry of the Riemannian-field $\mathbf{G}$ that

$$\frac{1}{2}A\langle D, D\rangle = \langle \nabla_A D, D\rangle \quad \text{for all} \quad A, D \in \mathfrak{X}(\mathrm{T}\mathcal{M}). \tag{54.3}$$

And hence

$$\begin{aligned}\langle\!\langle X, Y, Z, Z\rangle\!\rangle &= \frac{1}{2}Y(X\langle Z, Z\rangle) - \frac{1}{2}X(Y\langle Z, Z\rangle) + \frac{1}{2}[X, Y]\langle Z, Z\rangle \\ &= -\frac{1}{2}[X, Y]\langle Z, Z\rangle + \frac{1}{2}[X, Y]\langle Z, Z\rangle = 0.\end{aligned}$$

Since $X, Y, Z \in \mathfrak{X}(\mathrm{T}\mathcal{M})$ were arbitrary, the equation (I.2) follows. ▮

Let $\mathbf{C}$ be a compatible connection with the Riemannian-field $\mathbf{G}$.

Given $x \in \mathcal{M}$. Since $\mathbf{R}_x(\mathbf{C}) \in \mathrm{Skw}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{Lin}\,\mathrm{T}_x\mathcal{M})$, we observe form Prop. 2 that

$$\langle\!\langle \cdot\,, \cdot\,, \cdot\,, \cdot \rangle\!\rangle \in \mathrm{Skw}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{Skw}_2(\mathrm{T}_x\mathcal{M}^2,)).$$

Lemma : *Let an inner-product space $\mathcal{T}$, with $\dim \mathcal{T} \geq 2$, and a two-dimensional subspace $\mathcal{S}$ of $\mathcal{T}$ be given. If both $\{\mathbf{u}, \mathbf{v}\}$ and $\{\mathbf{s}, \mathbf{t}\}$ are bases for $\mathcal{S}$, then we have*

$$\frac{\mathbf{W}(\mathbf{u}, \mathbf{v}, \mathbf{u}, \mathbf{v})}{(\mathbf{u} \wedge \mathbf{v})(\mathbf{u}, \mathbf{v})} = \frac{\mathbf{W}(\mathbf{s}, \mathbf{t}, \mathbf{s}, \mathbf{t})}{(\mathbf{s} \wedge \mathbf{t})(\mathbf{s}, \mathbf{t})} \tag{54.4}$$

for all $\mathbf{W} \in \mathrm{Skw}_2(\mathcal{T}^2, \mathrm{Skw}_2(\mathcal{T}^2,))$.

Proof: By calculations. ▮

Applying the above Lemma, we arrive the following definition.

Definition : *Let $\mathcal{V} \subset \mathrm{T}_x\mathcal{M}$ be a two-dimensional subspace of $\mathrm{T}_x\mathcal{M}$. Let $\{\mathbf{u}, \mathbf{v}\}$ be a basis for $\mathcal{S}$. The* **sectional curvature of $\mathcal{S}$ at** *x is defined by*

$$\mathbf{K}_x(\mathcal{S}) := \frac{\langle\!\langle \mathbf{u}, \mathbf{v}, \mathbf{u}, \mathbf{v}\rangle\!\rangle}{(\mathbf{u} \wedge \mathbf{v})(\mathbf{u}, \mathbf{v})} \tag{54.5}$$

which does not depend on the choice of $\{\mathbf{u}, \mathbf{v}\}$.

Remark : The definition of sectional curvature "*does not*" require the assuption of the compatible connection $\mathbf{C}$ to be torsion-free. ▮

Proposition 4: *Let* $\mathbf{C}$ *be a connection on a Riemannian manifold* $\mathcal{M}$ *which is compatible with the Riemannian-field* $\mathbf{G}$*, then we have*

$$\langle\!\langle X, Y, Z, W \rangle\!\rangle - \langle\!\langle Z, W, X, Y \rangle\!\rangle = \mathbf{V}(X, Y, Z, W) \tag{54.6}$$

for all $X, Y, Z, W \in \mathfrak{X}(\mathrm{T}\mathcal{M})$.

Proof:

$$\begin{aligned}
&\mathbf{R}(X,Y)Z\cdot W + \mathbf{R}(Y,Z)X\cdot W + \mathbf{R}(Z,X)Y\cdot W\\
&\quad + \mathbf{R}(Y,Z)W\cdot X + \mathbf{R}(Z,W)Y\cdot X + \mathbf{R}(W,Y)Z\cdot X\\
&\quad + \mathbf{R}(Z,W)X\cdot Y + \mathbf{R}(W,X)Z\cdot Y + \mathbf{R}(X,Z)W\cdot Y\\
&\quad + \mathbf{R}(W,X)Y\cdot Z + \mathbf{R}(X,Y)W\cdot Z + \mathbf{R}(Y,W)X\cdot Z\\
&= \nabla\mathbf{T}(X,Y,Z)\cdot W + \nabla\mathbf{T}(Y,Z,X)\cdot W + \nabla\mathbf{T}(Z,X,Y)\cdot W\\
&\quad + \nabla\mathbf{T}(Y,Z,W)\cdot X + \nabla\mathbf{T}(Z,W,Y)\cdot X + \nabla\mathbf{T}(W,Y,Z)\cdot X\\
&\quad + \nabla\mathbf{T}(Z,W,X)\cdot Y + \nabla\mathbf{T}(W,X,Z)\cdot Y + \nabla\mathbf{T}(X,W,Z)\cdot Y\\
&\quad + \nabla\mathbf{T}(W,X,Y)\cdot Z + \nabla\mathbf{T}(X,Y,W)\cdot Z + \nabla\mathbf{T}(Y,W,X)\cdot Z\\
&\quad + \mathbf{T}(\mathbf{T}(X,Y),Z)\cdot W + \mathbf{T}(\mathbf{T}(Y,Z),X)\cdot W + \mathbf{T}(\mathbf{T}(Z,X),Y)\cdot W\\
&\quad + \mathbf{T}(\mathbf{T}(Y,Z),W)\cdot X + \mathbf{T}(\mathbf{T}(Z,W),Y)\cdot X + \mathbf{T}(\mathbf{T}(W,Y),Z)\cdot X\\
&\quad + \mathbf{T}(\mathbf{T}(Z,W),X)\cdot Y + \mathbf{T}(\mathbf{T}(W,X),Z)\cdot Y + \mathbf{T}(\mathbf{T}(X,Z),W)\cdot Y\\
&\quad + \mathbf{T}(\mathbf{T}(W,X),Y)\cdot Z + \mathbf{T}(\mathbf{T}(X,Y),W)\cdot Z + \mathbf{T}(\mathbf{T}(Y,W),X)\cdot Z
\end{aligned}$$

Proposition 5: *Let* $\mathbf{C}$ *be a connection on a Riemannian manifold* $\mathcal{M}$ *which is compatible with the Riemannian-field* $\mathbf{G}$*, then we have*

$$\mathrm{tr}\Big(\mathbf{R}(x)(\mathbf{s},\cdot)\,\mathbf{t} - \mathbf{R}(x)(\mathbf{t},\cdot)\,\mathbf{s} + \mathbf{R}(x)(\mathbf{t},\mathbf{s})\Big) = ???? \tag{54.7}$$

for all $\mathbf{s}, \mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

Second Proof of Pro. 2:

In view of (I.1) we have, for all $X, Y, Z, T \in \mathfrak{X}(\mathrm{T}\mathcal{M})$,

$$\langle \nabla_Y \nabla_X Z, T\rangle = Y\langle \nabla_X Z, T\rangle - \langle \nabla_X Z, \nabla_Y T\rangle,$$

$$\langle \nabla_X \nabla_Y Z, T\rangle = X\langle \nabla_Y Z, T\rangle - \langle \nabla_Y Z, \nabla_X T\rangle$$

and

$$\langle \nabla_{[X,Y]} Z, T\rangle = [X,Y]\langle Z,T\rangle - \langle Z, \nabla_{[X,Y]} T\rangle.$$

Hence

$$\begin{aligned}
\langle\!\langle X,Y,Z,T\rangle\!\rangle &= \langle \nabla_Y \nabla_X Z, T\rangle - \langle \nabla_X \nabla_Y Z, T\rangle + \langle \nabla_{[X,Y]} Z, T\rangle \\
&= Y\langle \nabla_X Z, T\rangle - \langle \nabla_X Z, \nabla_Y T\rangle - X\langle \nabla_Y Z, T\rangle + \langle \nabla_Y Z, \nabla_X T\rangle \\
&\quad + [X,Y]\langle Z,T\rangle - \langle Z, \nabla_{[X,Y]} T\rangle \\
&= Y(X\langle Z,T\rangle) - Y\langle Z, \nabla_X T\rangle - X(Y\langle Z,T\rangle) + X\langle Z, \nabla_Y T\rangle \\
&\quad - \langle \nabla_X Z, \nabla_Y T\rangle + \langle \nabla_Y Z, \nabla_X T\rangle + [X,Y]\langle Z,T\rangle - \langle Z, \nabla_{[X,Y]} T\rangle \\
&= -Y\langle Z, \nabla_X T\rangle + X\langle Z, \nabla_Y T\rangle \\
&\quad - \langle \nabla_X Z, \nabla_Y T\rangle + \langle \nabla_Y Z, \nabla_X T\rangle - \langle Z, \nabla_{[X,Y]} T\rangle \\
&= -\langle \nabla_Y \nabla_X T, Z\rangle + \langle \nabla_X \nabla_Y T, Z\rangle - \langle \nabla_{[X,Y]} T, Z\rangle \\
&= -\langle\!\langle X,Y,T,Z\rangle\!\rangle\,.
\end{aligned}$$

Since $X,Y,Z,T \in \mathfrak{X}(\mathrm{T}\mathcal{M})$ was arbitrary, the assertion of Prop. 2 follows.

55. Einstein-tensor field

Let a C^r manifold $\mathcal{M}$, with $r \geq 2$ and dim $\mathcal{M} \geq 2$, and a C^r connection $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\,(\mathrm{T}\mathcal{M})$ be given. Assume that $\mathbf{G} : \mathcal{M} \to \mathrm{Sym}_2(\mathrm{T}\mathcal{M}^2,)$ be a Riemannian-field compatiable with the connection $\mathbf{C}$.

Let $x \in \mathcal{M}$ be given and assume that the following condition hold

$$\mathrm{tr}\Big(\mathbf{R}(x)(\mathbf{s},\cdot)\,\mathbf{t} - \mathbf{R}(x)(\mathbf{t},\cdot)\,\mathbf{s} + \mathbf{R}(x)(\mathbf{t},\mathbf{s})\Big) = 0, \tag{55.1}$$

i.e. we have

$$\mathrm{tr}\,(\mathbf{R}(x)(\mathbf{s},\cdot)\,\mathbf{t}) - \mathrm{tr}\,(\mathbf{R}(x)(\mathbf{t},\cdot)\,\mathbf{s}) + \mathrm{tr}\,(\mathbf{R}(x)(\mathbf{t},\mathbf{s})) = 0.$$

Since $\mathbf{R}(x)(\mathbf{t},\mathbf{s})$ is skew-symmetric with respect to $\mathbf{G}$, we obtain that

$$\mathrm{tr}\,(\mathbf{R}(x)(\mathbf{s},\cdot)\,\mathbf{t}) = \mathrm{tr}\,(\mathbf{R}(x)(\mathbf{t},\cdot)\,\mathbf{s}) \qquad \text{for all} \quad \mathbf{s},\mathbf{t} \in \mathrm{T}_x\mathcal{M}.$$

<u>Definition</u> : *The* **Ricci-tensor field** $\mathrm{Ric} : \mathcal{M} \to \mathrm{Sym}_2(\mathrm{T}\mathcal{M}^2,)$ *is defined by*

$$\mathrm{Ric}(x)(\mathbf{s},\mathbf{t}) := \mathrm{tr}\,(\mathbf{R}(x)(\mathbf{s},\cdot)\,\mathbf{t}) \tag{55.2}$$

for all $x \in \mathcal{M}$ *and all* $\mathbf{s},\mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

<u>Definition</u> : *The* **Einstein-tensor field** $\mathrm{Ein} : \mathcal{M} \to \mathrm{Sym}_2(\mathrm{T}\mathcal{M}^2,)$ *is defined by*

$$\mathrm{Ein}(x) := \mathrm{Ric}(x) - \frac{1}{2}\mathrm{tr}\,(\mathbf{G}^{-1}(x)\mathrm{Ric}(x))\,\mathbf{G}(x) \tag{55.3}$$

for all $x \in \mathcal{M}$. (The factor 1/2 is determined by the assumption $\dim \mathrm{T}_x\mathcal{M} = 4$!)

It follows from the 2nd Bianchi Identity (this condition should be weaken) that

$$\mathrm{div}_{\mathbf{C}}\,\mathrm{Ein} = 0. \tag{55.4}$$

Remark: The **matter tensor field** $\mathrm{Mat} : \mathcal{M} \to \mathrm{Sym}_2(\mathrm{T}\mathcal{M}^2,)$ satisfying

$$\mathrm{Ein}(x) = \kappa\,\mathrm{Mat}(x) \tag{55.5}$$

where $\kappa \in$ is the **universal gravitational constant**. It follows from (Ein.4) and (Ein 5) that

$$\mathrm{div}_{\mathbf{C}}\,\mathrm{Mat} = 0 \tag{55.6}$$

(balance of world-momentum). ∎