

The Theory of Surface Interactions

by Walter Noll, February 2005 *

0. Introduction

In 1823 Cauchy introduced the concept of the *stress tensor*, which has been of central importance in all branches of continuum mechanics ever since. In 1957 I was familiar with the textbook proofs of the existence of the stress tensor, based on what is usually called the "Cauchy Stress Principle". I then proved that this Principle can itself be derived from more basic hypotheses (see reference [N1]). Abstract versions of these two results are given as Theorems B and A in Section 6 below.

Very little research on these matters was done between 1957 and 1967. Then a number of authors reinvestigated these issues in various forms. These are their names, in alphabetical order:

Fosdick, Gurtin, Martins, Mizel, Šilhavý, Virga, Williams, and Ziemer.
(There may be others that I don't know about.)

Most of this literature deals with contact forces or heat transfers. Here I introduce the general concept of *interaction*, which is an abstraction that can be applied to force systems, torque systems, heat-transfer systems, entropy-transfer systems, etc. The concept of a *contact* introduced in Sect.5 is an abstraction that becomes the stress tensor in the case of force systems, the couple-stress tensor in the case of torque systems, and the heat flux vector in the case heat-transfer systems.

The treatment given here differs from all previous ones by the use of the concept of a *fit region* as defined and analyzed in reference [NV1]. The regions occupied by a continuous body and its parts are assumed to be fit regions. The concepts of the *reduced contact* and *directed contact* of two such regions, as defined in Sect.4 below, will be of central importance.

Some of the ideas presented here were discussed in previous papers, in particular in [N2] and [NV1]. The proofs of most of the statements in Sects.1-4 are given in these papers and will not be repeated here.

Note: We use the terminology and notation of [N3]. In particular, we use the following abbreviations:

\mathbb{P}^{\times} := the set of all strictly positive real numbers, Dom:= domain,
Clo:= closure, Int:= interior, Bdy:= boundary, Ubl:= unit ball,
Usph:= unit sphere, Lin:= space of linear mappings, vol:= volume, ar:= area.

* This paper is based on the lecture that I gave at the meeting of the Society for Natural Philosophy in April 1993, which took place at Carnegie Mellon University on the occasion of my retirement from teaching .

1. Materially ordered sets

Let a set Ω ordered by \prec be given. We use the notations

$$\mathcal{P} \wedge \mathcal{Q} := \inf\{\mathcal{P}, \mathcal{Q}\}, \quad \mathcal{P} \vee \mathcal{Q} := \sup\{\mathcal{P}, \mathcal{Q}\} \quad \text{for all } \mathcal{P}, \mathcal{Q} \in \Omega. \quad (1.1)$$

Definition 1. We say that Ω is **materially ordered** by \prec if:

- (M₁) Ω has a maximum ∞_Ω and a minimum \emptyset_Ω .
- (M₂) For every $\mathcal{P} \in \Omega$ there is exactly one element of Ω , denoted by \mathcal{P}^e and called the **exterior** of \mathcal{P} , such that

$$\mathcal{P} \wedge \mathcal{P}^e = \emptyset_\Omega \quad \text{and} \quad \mathcal{P} \vee \mathcal{P}^e = \infty_\Omega. \quad (1.2)$$

- (M₃) For all $\mathcal{P}, \mathcal{Q} \in \Omega$ we have

$$\mathcal{P} \wedge \mathcal{Q}^e = \emptyset_\Omega \implies \mathcal{P} \prec \mathcal{Q}. \quad (1.3)$$

- (M₄) $\mathcal{P} \wedge \mathcal{Q}$ exists for all $\mathcal{P}, \mathcal{Q} \in \Omega$.

The following two properties are consequences of (M₁) - (M₄):

- (M₅) For all $\mathcal{P}, \mathcal{Q} \in \Omega$ we have

$$\mathcal{P} \wedge \mathcal{Q}^e = \emptyset_\Omega \iff \mathcal{P} \prec \mathcal{Q}. \quad (1.4)$$

- (M₄) $\mathcal{P} \vee \mathcal{Q}$ exists for all $\mathcal{P}, \mathcal{Q} \in \Omega$ and

$$\mathcal{P} \vee \mathcal{Q} = (\mathcal{P}^e \wedge \mathcal{Q}^e)^e. \quad (1.5)$$

Theorem 1. *If Ω is materially ordered, it acquires the structure of a Boolean algebra relative to $\vee, \wedge, ()^e, \emptyset_\Omega, \infty_\Omega$.*

2. Interactions

We assume that a materially ordered set Ω and a linear space \mathcal{W} are given. We use the **notation**:

$$(\Omega^2)_{\text{sep}} := \{(\mathcal{P}, \mathcal{Q}) \in \Omega^2 \mid \mathcal{P} \wedge \mathcal{Q} = \emptyset_\Omega\}, \quad (2.1)$$

and call its members **separate pairs**. Given $\mathcal{P} \in \Omega$ we put

$$\Omega_{\mathcal{P}} := \{\mathcal{R} \in \Omega \mid \mathcal{P} \prec \mathcal{Q}\}. \quad (2.2)$$

The set $\Omega_{\mathcal{P}}$ is again materially ordered, with \mathcal{P} as maximum and $\mathcal{R} \mapsto \mathcal{R}^e \wedge \mathcal{P}$ as its exterior formation.

Definition 2. A function $H : \Omega \longrightarrow \mathcal{W}$ is said to be **additive** if

$$H(\mathcal{P} \vee \mathcal{Q}) = H(\mathcal{P}) + H(\mathcal{Q}) \quad \text{for all } (\mathcal{P}, \mathcal{Q}) \in (\Omega^2)_{\text{sep}} . \quad (2.3)$$

A function $I : (\Omega^2)_{\text{sep}} \longrightarrow \mathcal{W}$ is called an **interaction** in Ω if, for all $\mathcal{P} \in \Omega$, both $I(\cdot, \mathcal{P}^e) : \Omega_{\mathcal{P}} \longrightarrow \mathcal{W}$ and $I(\mathcal{P}^e, \cdot) : \Omega_{\mathcal{P}} \longrightarrow \mathcal{W}$ are additive.

The **resultant** $R_I : \Omega \longrightarrow \mathcal{W}$ of a given interaction I in Ω is defined by

$$R_I(\mathcal{P}) := I(\mathcal{P}, \mathcal{P}^e) . \quad (2.4)$$

We say that the given interaction is **skew** (or "obeys the law of action and reaction") if

$$I(\mathcal{Q}, \mathcal{P}) = -I(\mathcal{P}, \mathcal{Q}) \quad \text{for all } (\mathcal{P}, \mathcal{Q}) \in (\Omega^2)_{\text{sep}} . \quad (2.5)$$

Theorem 2. An interaction is skew if and only if its resultant is additive.

3. Fit regions.

We consider the following problem: What type of subsets of Euclidean spaces are "fit" to be "regions occupied by a continuous body"?

After some thought, I came to the conclusion, long ago, that the class of all such fit regions should satisfy the following **requirements**:

A fit region should

- (a) be an *open set* (domain of C^1 - mappings),
- (b) have a boundary that is also the boundary of its exterior (no "hidden" boundary points),
- (c) have an *exterior normal* at all but "exceptional" boundary points, and the Integral-Gradient Theorem (Divergence Theorem) should be applicable.
- (d) The class of all fit regions should be invariant under whole-space C^2 -diffeomorphisms.
- (e) The fit regions included in a given one should be *materially ordered* by inclusion.

We consider the following classes:

Ro : regularly open sets. (does not satisfy (c))

Bnb : bounded sets with negligible boundary. (does not satisfy (a),(b),(c))

Fp : sets of finite perimeter. (does not satisfy (a), (b))

Po : polyhedral sets. (does not satisfy (d))

Reg : regular regions as defined in Sect.1 of [NV2]. (does not satisfy (e))

(A set is *regularly open* if it is the interior of its closure. A subset of a Euclidean space is *negligible* if it can be covered by a finite collection of balls with arbitrarily small total volume. Sets of *finite perimeter* are defined in Def.4 on p.12 of [NV1].)

Ideally, the class of all fit regions should include all that can possibly imagined by an engineer but exclude those that only an ingenious mathematician can think of. It is about 30 years ago that I reached the conclusion that there is no such class. I now believe that the best one can do is use the class of **fit regions** proposed in [NV1]:

$$\text{Fr} = \text{Ro} \cap \text{Bnb} \cap \text{Fp} , \quad (3.1)$$

which satisfies all the requirements.

Theorem 3. *Let $\mathcal{B} \in \text{Fr}$ be given and put*

$$\Omega := \text{Fr } \mathcal{B} := \{ \mathcal{P} \in \text{Fr} \mid \mathcal{P} \subset \mathcal{B} \} . \quad (3.2)$$

*Then Ω is materially ordered by inclusion with $\emptyset_\Omega := \emptyset$, $\infty_\Omega := \mathcal{B}$. In this case, we use the term **exterior relative to \mathcal{B}** and denote it by $()^\mathcal{B}$ rather than simply $()^e$. For all $\mathcal{P}, \mathcal{Q} \in \Omega$ we have*

$$\mathcal{P}^\mathcal{B} = \text{Int}(\mathcal{B} \setminus \mathcal{P}) , \quad (3.3)$$

$$\mathcal{P} \wedge \mathcal{Q} = \mathcal{P} \cap \mathcal{Q} , \quad (3.4)$$

$$\mathcal{P} \vee \mathcal{Q} = \text{Int} \text{ Clo}(\mathcal{P} \cup \mathcal{Q}) . \quad (3.5)$$

4. Reduced boundaries.

We assume that a Euclidean space \mathcal{E} with translation space \mathcal{V} and a fit region $\mathcal{P} \in \text{Fr } \mathcal{E}$ are given. Note that the set $x + r\text{Ubl } \mathcal{V}$ is the open ball of radius $r \in \mathbb{P}^\times$ and center $x \in \mathcal{E}$.

Definition 3. *The density function*

$$d_{\mathcal{P}} : \mathcal{E} \longrightarrow [0, 1] \cup \{T\} \quad (4.1)$$

(read T as "trash") of \mathcal{P} is defined by

$$d_{\mathcal{P}}(x) := \begin{cases} \lim_{r \rightarrow 0} \frac{\text{vol}(\mathcal{P} \cap (x + r\text{Ubl } \mathcal{V}))}{\text{vol}(x + r\text{Ubl } \mathcal{V})} & \text{if the limit exists} \\ T & \text{otherwise.} \end{cases} \quad (4.2)$$

The essential interior and the essential boundary of \mathcal{P} are defined by

$$\text{Int } *_\mathcal{P} := \{x \in \mathcal{E} \mid d_{\mathcal{P}}(x) = 1\} , \quad (4.3)$$

$$\text{Bdy } *_\mathcal{P} := \{x \in \mathcal{E} \mid d_{\mathcal{P}}(x) \notin \{0, 1\}\} , \quad (4.4)$$

respectively.

Theorem 4. *We have*

$$\text{vol}(\text{Bdy } *_\mathcal{P}) = 0, \quad \text{Int } \mathcal{P} \subset \text{Int } *_\mathcal{P}, \quad \text{Bdy } *_\mathcal{P} \subset \text{Bdy } \mathcal{P} . \quad (4.5)$$

Given $\mathbf{u} \in \text{Usph } \mathcal{V}$, the **open half-space** in \mathcal{V} determined by \mathbf{u} is defined by

$$H(\mathbf{u}) := \mathbb{P}^\times \mathbf{u} + \{\mathbf{u}\}^\perp \quad (4.6)$$

Definition 4. The **half-density function**

$$d_{\mathbf{u}, \mathcal{P}} : \longrightarrow [0, \frac{1}{2}] \cup \{T\} \quad (4.7)$$

of \mathcal{P} relative to \mathbf{u} is defined by

$$d_{\mathbf{u}, \mathcal{P}}(x) := d_{\mathcal{P} \cap (x + H(\mathbf{u}))}(x) \quad \text{for all } x \in \mathcal{E}. \quad (4.8)$$

(see Fig.1)

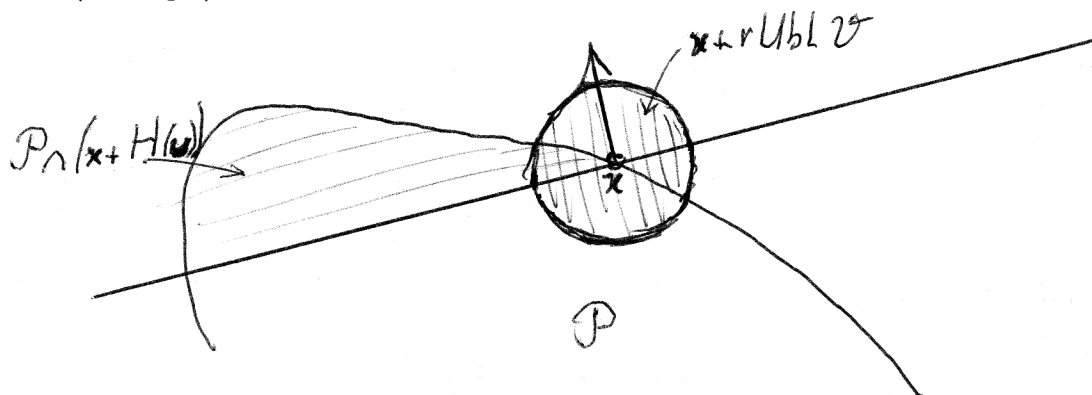


Fig.1

Proposition 1. Given $x \in \mathcal{E}$, the problem

$$? \mathbf{u} \in \text{Usph } \mathcal{V} \text{ such that } d_{\mathbf{u}, \mathcal{P}}(x) = 0, \quad d_{-\mathbf{u}, \mathcal{P}}(x) = \frac{1}{2} \quad (4.9)$$

has at most one solution.

Definition 5. The **reduced boundary** of \mathcal{P} is defined by

$$\text{Rby } \mathcal{P} := \{x \in \mathcal{E} \mid (4.9) \text{ has a solution}\} \quad (4.10)$$

and the **oriented boundary**

$$\text{no } \mathcal{P} : \text{Rby } \mathcal{P} \longrightarrow \text{Usph } \mathcal{V} \quad (4.11)$$

of \mathcal{P} is defined by

$$\text{no } \mathcal{P}(x) := \text{the solution of } (4.9) \quad (4.12)$$

Roughly speaking, the reduced boundary is obtained from the essential boundary by omitting all corners, edges, cusps, etc. and including only points x at which a tangent plane can be defined, with $\text{no } \mathcal{P}(x)$ the unit vector normal to it and directed away from \mathcal{P} .

We have $\text{Clo Rby } \mathcal{P} = \text{Bdy } \mathcal{P}$.

For fit regions, one can define an area-measure on the reduced boundary and the following is valid (see Sect.6 of [NV1]):

Integral-Gradient Theorem. Let $\mathcal{P} \in \text{Fr } \mathcal{E}$, a finite-dimensional linear space \mathcal{W} , an open subset \mathcal{B} of \mathcal{E} with $\text{Clo } \mathcal{P} \subset \mathcal{B}$ and a C^1 -function $\mathbf{h} : \mathcal{B} \rightarrow \mathcal{W}$ be given. Then

$$\int_{\mathcal{P}} \nabla \mathbf{h} \, \text{dvol} = \int_{\text{Rby } \mathcal{P}} \mathbf{h} \otimes \text{no } \mathcal{P} \, \text{dar} . \quad (4.13)$$

If $\mathcal{W} := \mathcal{V}$ then one can take the value-wise trace of this formula and obtain the more familiar *Divergence Theorem*:

$$\int_{\mathcal{P}} \text{div} \mathbf{h} \, \text{dvol} = \int_{\text{Rby } \mathcal{P}} \mathbf{h} \cdot \text{no } \mathcal{P} \, \text{dar} . \quad (4.14)$$

Definition 6. Let $\mathcal{P}, \mathcal{Q} \in \text{Fr } \mathcal{E}$ with $\mathcal{P} \cap \mathcal{Q} = \emptyset$ be given. Then the **reduced contact** of \mathcal{P}, \mathcal{Q} is defined by

$$\text{Rct}(\mathcal{P}, \mathcal{Q}) := \text{Rby } \mathcal{P} \cap \text{Rby } \mathcal{Q} . \quad (4.15)$$

The **directed contact** from \mathcal{P} to \mathcal{Q} is defined by

$$\text{no}(\mathcal{P}, \mathcal{Q}) := \text{no } \mathcal{P} |_{\text{Rct}(\mathcal{P}, \mathcal{Q})} . \quad (4.16)$$

If $\text{Rct}(\mathcal{P}, \mathcal{Q}) = \emptyset$, then $\text{no}(\mathcal{P}, \mathcal{Q})$ is the empty mapping, also denoted by \emptyset . It is clear that we have

$$\text{no}(\mathcal{P}, \mathcal{Q}) = -\text{no}(\mathcal{Q}, \mathcal{P}) . \quad (4.17)$$

Proposition 2. Let $\mathcal{P}, \mathcal{P}', \mathcal{Q}, \mathcal{Q}' \in \text{Fr } \mathcal{E}$ with $\mathcal{P} \cap \mathcal{Q} = \emptyset$ and $\mathcal{P}' \cap \mathcal{Q}' = \emptyset$ be given such that $\text{no}(\mathcal{P}, \mathcal{Q}) = \text{no}(\mathcal{P}', \mathcal{Q}')$. Then $(\mathcal{P} \cap \mathcal{P}') \cap (\mathcal{Q} \cap \mathcal{Q}') = \emptyset$ and

$$\text{no}(\mathcal{P}, \mathcal{Q}) = \text{no}_{(\mathcal{P} \cap \mathcal{P}') \cap (\mathcal{Q} \cap \mathcal{Q}')} = \text{no}(\mathcal{P}', \mathcal{Q}') . \quad (4.18)$$

Note: The concept of a directed contact generalizes the concept of an *oriented surface*.

We assume now that $\mathcal{B} \in \text{Fr } \mathcal{E}$ is given and we consider the materially ordered set $\Omega := \text{Fr } \mathcal{B}$ as in (3.2).

Definition 7. The set of all **directed contacts** for Ω is defined by

$$\text{Dc}_\Omega := \{ \text{no}(\mathcal{P}, \mathcal{Q}) \mid (\mathcal{P}, \mathcal{Q}) \in (\Omega^2)_{\text{sep}} \} . \quad (4.19)$$

Notation: For functions with codomain $\text{Usph } \mathcal{V}$, we understand inclusion, union, and intersection in terms of their graphs. Thus:

$$\mathbf{n} \subset \mathbf{m} \iff (\text{Dom } \mathbf{n} \subset \text{Dom } \mathbf{m} \text{ and } \mathbf{n} = \mathbf{m}|_{\text{Dom } \mathbf{n}}) \text{ for all } \mathbf{n}, \mathbf{m} \in \text{Dc} . \quad (4.20)$$

Let $\mathbf{n} \in \text{Dc}_\Omega$ be given. We denote the set of all directed contacts included in \mathbf{n} by

$$\text{Dc}_\mathbf{n} := \{ \mathbf{m} \in \text{Dc}_\Omega \mid \mathbf{m} \subset \mathbf{n} \}. \quad (4.21)$$

Of course, for every $\mathbf{n}' \in \text{Dc}_\mathbf{n}$, we have $\text{Dc}_{\mathbf{n}'} \subset \text{Dc}_\mathbf{n}$.

Choosing $(\mathcal{P}, \mathcal{Q}) \in (\Omega^2)_{\text{sep}}$ such that $\mathbf{n} = \text{no}_{(\mathcal{P}, \mathcal{Q})}$, we have

$$\text{Dc}_\mathbf{n} = \{ \text{no}_{(\mathcal{R}, \mathcal{Q})} \mid \mathcal{R} \in \text{Fr } \mathcal{B}, \mathcal{R} \subset \mathcal{P} \}. \quad (4.22)$$

Theorem 5. *The order, by inclusion, of the set $\text{Dc}_\mathbf{n}$ is a material order in the sense of Def.1. The minimum is the empty set and the maximum is \mathbf{n} .*

Let $\mathbf{m}, \mathbf{m}' \in \text{Dc}_\mathbf{n}$ be given and put $\mathcal{N} := \text{Dom } \mathbf{n}$, $\mathcal{M} := \text{Dom } \mathbf{m}$, $\mathcal{M}' := \text{Dom } \mathbf{m}'$. Then the exterior \mathbf{m}^e of \mathbf{m} is given by

$$\mathbf{m}^e = \mathbf{n}|_{\text{Dom } \mathbf{m}^e} \quad (4.23)$$

with

$$\text{Dom } \mathbf{m}^e = (\mathcal{N} \setminus \mathcal{M}) \setminus (\text{Clo } \mathcal{M} \cap \text{Clo } (\mathcal{N} \setminus \mathcal{M}) \cap \mathcal{N}). \quad (4.24)$$

The meet $\mathbf{m} \wedge \mathbf{m}'$ and the join $\mathbf{m} \vee \mathbf{m}'$ are given by

$$\mathbf{m} \wedge \mathbf{m}' = \mathbf{m} \cap \mathbf{m}' = \mathbf{n}|_{\mathcal{M} \cap \mathcal{M}'} \quad (4.25)$$

and

$$\mathbf{m} \vee \mathbf{m}' = \mathbf{n}|_{\text{Dom } (\mathbf{m} \wedge \mathbf{m}')} \quad (4.26)$$

with

$$\text{Dom } (\mathbf{m} \vee \mathbf{m}') = \mathcal{M} \cup \mathcal{M}' \cup (\text{Clo } \mathcal{M} \cap \text{Clo } \mathcal{M}' \cap \mathcal{N}). \quad (4.27)$$

Outline of proof: We choose $(\mathcal{P}, \mathcal{Q}) \in (\Omega^2)_{\text{sep}}$ such that $\mathbf{n} = \text{no}_{(\mathcal{P}, \mathcal{Q})}$. By (4.22) one can then determine \mathcal{R} and \mathcal{R}' in such a way that $\mathbf{m} = \text{no}_{(\mathcal{R}, \mathcal{Q})}$ and $\mathbf{m}' = \text{no}_{(\mathcal{R}', \mathcal{Q})}$ have appropriate properties. For example, for the meet $\mathbf{m} \wedge \mathbf{m}'$, see Fig.2, where the point z belongs to $(\text{Clo } \mathcal{M} \cap \text{Clo } \mathcal{M}' \cap \mathcal{N})$ but not to $\mathcal{M} \cup \mathcal{M}'$.

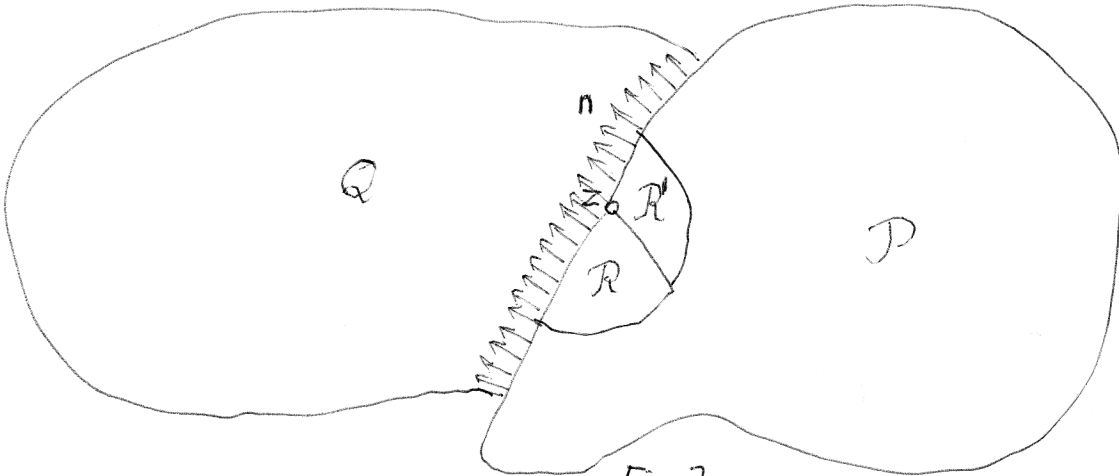


Fig. 2

Definition 8. We say that a given $\mathbf{n} \in \text{Dc}_\Omega$ is **plane**, and we write

$$\mathbf{n} := (\mathcal{S}, \mathbf{u}),$$

if \mathbf{n} is constant with value \mathbf{u} and if $\mathcal{S} := \text{Dom } \mathbf{n}$ is included in a plane perpendicular to \mathbf{u} .

In this case, the members of $\text{Dc}_\mathbf{n}$ are also all plane and their domains are subsets of \mathcal{S} .

5. Surface interactions

Again, we assume that $\mathcal{B} \in \text{Fr } \mathcal{E}$ is given and we consider the materially ordered set $\Omega := \text{Fr } \mathcal{B}$ as in Thm.3 .

Definition 9. We say that a given interaction I in Ω with values in a given linear space \mathcal{W} is a **surface interaction** if

$$\text{ar}(\text{Rct}(\mathcal{P}, \mathcal{Q})) = 0 \implies I(\mathcal{P}, \mathcal{Q}) = \mathbf{0} \quad \text{for all } (\mathcal{P}, \mathcal{Q}) \in (\Omega^2)_{\text{sep}}. \quad (5.1)$$

Note: Not every contact interaction, as defined in Sect.4 of [NV2], is a surface interaction. The edge interactions discussed in [NV2] are contact interactions but not surface interactions.

Theorem 6. For every surface interaction I there is exactly one function $\mathbf{C} : \text{Dc}_\Omega \longrightarrow \mathcal{W}$ such that

$$\mathbf{C}(\text{no}_{(\mathcal{P}, \mathcal{Q})}) = I(\mathcal{P}, \mathcal{Q}) \quad \text{for all } (\mathcal{P}, \mathcal{Q}) \in (\Omega^2)_{\text{sep}}. \quad (5.2)$$

Proof: Let $\mathbf{n} \in \text{Dc}_\Omega$ be given and choose $(\mathcal{P}', \mathcal{Q}') \in (\Omega^2)_{\text{sep}}$ such that $\mathbf{n} = \text{no}_{(\mathcal{P}', \mathcal{Q}')}$. Now let $(\mathcal{P}, \mathcal{Q}) \in (\Omega^2)_{\text{sep}}$ be given such that $\mathbf{n} = \text{no}_{(\mathcal{P}, \mathcal{Q})}$. Put $\mathcal{P}'' := \text{Int}(\mathcal{P} \setminus (\mathcal{P} \cap \mathcal{P}'))$. Then $\mathcal{P} = \mathcal{P}'' \vee (\mathcal{P} \cap \mathcal{P}')$ and $(\mathcal{P}'', (\mathcal{P} \cap \mathcal{P}')) \in (\Omega^2)_{\text{sep}}$. (See Fig.3 below.) Hence, by the additivity of $I(\cdot, \mathcal{Q})$ we have

$$I(\mathcal{P}, \mathcal{Q}) = I(\mathcal{P} \cap \mathcal{P}', \mathcal{Q}) + I(\mathcal{P}'', \mathcal{Q}). \quad (5.3)$$

It is easily seen that

$$\text{no}_{(\mathcal{P}, \mathcal{Q})} = \text{no}_{(\mathcal{P} \cap \mathcal{P}', \mathcal{Q})} = \text{no}_{(\mathcal{P}', \mathcal{Q})} \quad (5.4)$$

and $\text{Rct}(\mathcal{P}'', \mathcal{Q}) = \emptyset$. Hence, Since the area of the empty set is zero, it follows from (5.1) that $I(\mathcal{P}'', \mathcal{Q}) = 0$ and hence

$$I(\mathcal{P}, \mathcal{Q}) = I(\mathcal{P} \cap \mathcal{P}', \mathcal{Q}). \quad (5.5)$$

Interchanging the roles of \mathcal{P} and \mathcal{P}' , we see that

$$I(\mathcal{P}', \mathcal{Q}) = I(\mathcal{P} \cap \mathcal{P}', \mathcal{Q}). \quad (5.6)$$

and hence $I(\mathcal{P}', \mathcal{Q}) = I(\mathcal{P}, \mathcal{Q})$.

A similar argument shows that $I(\mathcal{P}', \mathcal{Q}) = I(\mathcal{P}', \mathcal{Q}')$. Using (5.5) and (5.6), it follows that $I(\mathcal{P}', \mathcal{Q}') = I(\mathcal{P}, \mathcal{Q})$ and hence that $I(\mathcal{P}, \mathcal{Q})$ depends only on $\mathbf{n} = \text{no}(\mathcal{P}', \mathcal{Q}')$. ■

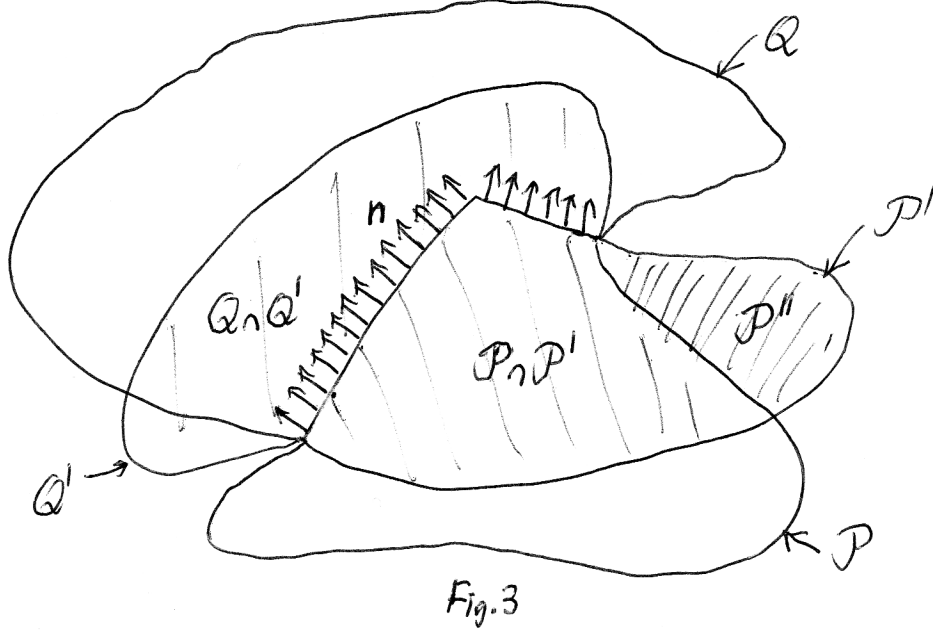


Fig. 3

We say that \mathbf{C} is the **contact flux** associated with I and we have

$$\text{ar}(\text{Dom } \mathbf{n}) = 0 \implies \mathbf{C}(\mathbf{n}) = \mathbf{0} \quad \text{for all } \mathbf{n} \in \text{Dc}_\Omega. \quad (5.7)$$

Theorem 7. Let $\mathbf{n} \in \text{Dc}_\Omega$ be given. Then the restriction

$$\mathbf{C}_\mathbf{n} := \mathbf{C}|_{\text{Dc}_\mathbf{n}} \quad (5.8)$$

of the contact flux \mathbf{C} to the materially ordered set $\text{Dc}_\mathbf{n}$ is additive in the sense of Def. 2, i.e. we have

$$\mathbf{C}_\mathbf{n}(\mathbf{m} \wedge \mathbf{m}') = \mathbf{C}_\mathbf{n}(\mathbf{m}) + \mathbf{C}_\mathbf{n}(\mathbf{m}') \quad \text{for all } (\mathbf{m}, \mathbf{m}') \in (\text{Dc}_\mathbf{n})_{\text{sep}}^2. \quad (5.9)$$

Proposition 3. The surface interaction I is skew in the sense of (2.5) if and only if

$$\mathbf{C}(-\mathbf{n}) = -\mathbf{C}(\mathbf{n}) \quad \text{for all } \mathbf{n} \in \text{Dc}_\Omega. \quad (5.10)$$

6. Contactors and Proto-contactors.

Let a fit region \mathcal{B} and a surface interaction I for $\text{Fr } \mathcal{B}$ be given as in the previous section. Also, let a subset Γ of Dc_Ω be given.

Definition 10. We say that a function

$$\mathbf{s} : \mathcal{B} \times \text{Usph } \mathcal{V} \longrightarrow \mathcal{W} \quad (6.1)$$

is a **proto-contactor** of I relative to Γ if

$$\mathbf{C}(\mathbf{n}) = \int_{\text{Dom } \mathbf{n}} \mathbf{s}(x, \mathbf{n}(x)) \, \text{dar}_x \quad \text{for all } \mathbf{n} \in \Gamma . \quad (6.2)$$

Definition 11. We say that a function

$$\mathbf{S} : \mathcal{B} \longrightarrow \text{Lin}(\mathcal{V}, \mathcal{W}) \quad (6.3)$$

is a **contactor** of I relative to Γ if

$$((x, \mathbf{u}) \mapsto \mathbf{S}(x)\mathbf{u}) : \mathcal{B} \times \text{Usph } \mathcal{V} \longrightarrow \mathcal{W} \quad (6.4)$$

is a **proto-contactor** of I relative to Γ , so that

$$\mathbf{C}(\mathbf{n}) = \int_{\text{Dom } \mathbf{n}} \mathbf{S}(x)\mathbf{n}(x) \, \text{dar}_x \quad \text{for all } \mathbf{n} \in \Gamma . \quad (6.5)$$

In 1823, Cauchy proved, in essence, a theorem of the following form: If I admits a proto-contactor relative to a suitable class Γ and if suitable assumptions are made then I admits a contactor relative to Γ . Cauchy simply assumed the existence of a proto-contactor. This assumption is often called "the Cauchy Stress Principle". The description of the class Γ and the assumptions needed for the proof of his theorem were quite vague by modern standards.

In 1957 (see Theorem IV on p.275 in [N1]), I proved a theorem of the following form: If suitable assumptions are made then I admits a proto-contactor relative to an appropriate class Γ . Again, the description of the class Γ and the assumptions needed for the proof of my theorem were not very precise. One reason was that, at that time, the requisite conceptual infrastructure was missing.

We will now examine the nature of the conditions that would be appropriate to modern standards.

Conditions:

Given: A surface interaction I with corresponding contact-flux \mathbf{C} .

(a) The resultant R_I of I , as defined by (2.4), is **locally volume-bounded**. More precisely: For every $x \in \mathcal{B}$ there is $\mathcal{M} \in \Omega$ with $x \in \mathcal{M}$ and there is $k \in \mathbb{P}^\times$ such that

$$|R_I(\mathcal{P})| \leq k \, \text{vol}(\mathcal{P}) \quad \text{for all } \mathcal{P} \in \Omega_{\mathcal{M}} , \quad (6.6)$$

where $\Omega_{\mathcal{M}}$ is defined according to (2.2).

(b) I is **locally area-bounded**. More precisely: For every $x \in \mathcal{B}$ there is $\mathcal{M} \in \Omega$ with $x \in \mathcal{M}$ and there is $h \in \mathbb{P}^\times$ such that

$$|I(\mathcal{P}, \mathcal{Q})| \leq h \operatorname{ar}(\operatorname{Rct}(\mathcal{P}, \mathcal{Q})) \quad \text{for all } (\mathcal{P}, \mathcal{Q}) \in (\Omega_{\mathcal{M}}^2)_{\text{sep}}, \quad (6.7)$$

where $(\Omega_{\mathcal{M}}^2)_{\text{sep}}$ is defined according to (2.1).

(c) For every $\mathbf{n} \in \operatorname{Dc}_\Omega$ the additive function $\mathbf{C}_{\mathbf{n}} : \operatorname{Dc}_{\mathbf{n}} \rightarrow \mathcal{W}$ described in Theorem 7 has a continuous density $\mathbf{s}_{\mathbf{n}} : \operatorname{Dom} \mathbf{n} \rightarrow \mathcal{W}$, so that

$$\mathbf{C}_{\mathbf{n}}(\mathbf{m}) = \int_{\operatorname{Dom} \mathbf{m}} \mathbf{s}_{\mathbf{n}}(x) \operatorname{dar}_x \quad \text{for all } \mathbf{m} \in \operatorname{Dc}_{\mathbf{n}}. \quad (6.8)$$

It is easily seen that (c) implies (b). Adding some additional hypotheses to (b) may insure that a modified (b) implies (c), but it is not clear to me how such additional hypotheses should be formulated.

Theorem A. *If (a) and (c) hold, then I admits a proto-contactor relative to the class*

$$\operatorname{SDc}_\Omega := \{ \mathbf{n} \in \operatorname{Dc}_\Omega \mid \operatorname{Dom} \mathbf{n} \text{ is a } C^1 \text{ - surface} \}. \quad (6.9)$$

Proof: Let $x \in \mathcal{B}$ and $\mathbf{u} \in \operatorname{Usph} \mathcal{V}$ be given. The plane $x + \{\mathbf{u}\}^\perp$ through x and perpendicular to \mathbf{u} cuts \mathcal{B} into two parts, both of which belong to $\operatorname{Fr} \mathcal{B}$. The directed contact from one to the other of these two parts is a plane contact of the form $(\mathcal{S}, \mathbf{u})$, where \mathcal{S} is included in the plane $x + \{\mathbf{u}\}^\perp$. By (c), this plane contact has a density, whose value at x we denote by $\mathbf{s}(x, \mathbf{u})$. For every subset \mathcal{T} of \mathcal{S} such that $(\mathcal{T}, \mathbf{u})$ belongs to Dc_Ω , we then have

$$\mathbf{C}((\mathcal{T}, \mathbf{u})) = \int_{\mathcal{T}} \mathbf{s}(x, \mathbf{u}) \operatorname{dar}_x. \quad (6.10)$$

Now, for every $r \in \mathbb{P}^\times$, consider the open half-ball

$$\mathcal{C}_r := x + r \operatorname{Ubl} \mathcal{V} \cap H(-\mathbf{u}) \quad (6.11)$$

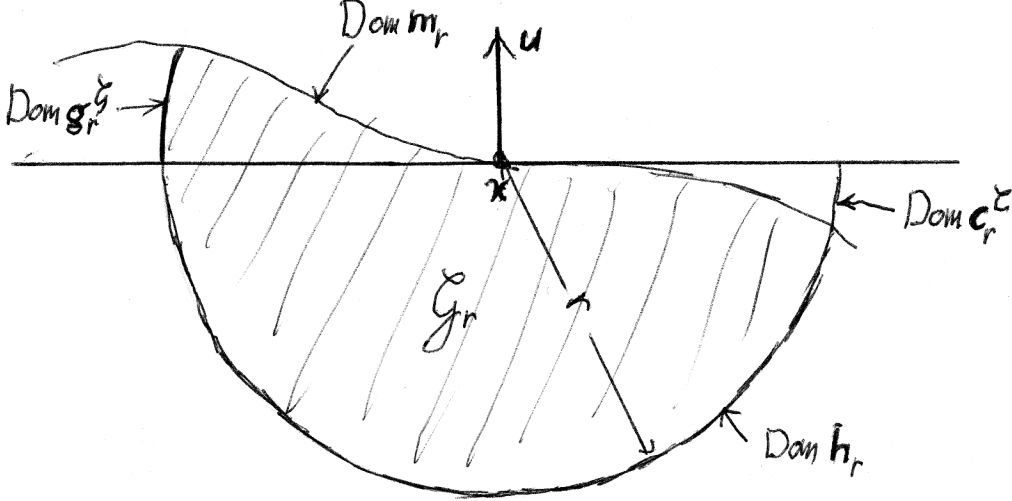
where $H(-\mathbf{u})$ is the open half-space in \mathcal{V} defined by $-\mathbf{u}$ (see (4.6)).

If r is small enough, then \mathcal{C}_r is included in \mathcal{B} and belongs to set $\Omega := \operatorname{Fr} \mathcal{B}$. The oriented boundary of \mathcal{C}_r is the union of two disjoint directed contacts. One of these is plane, namely the oriented upper disc $(\mathcal{D}_r, \mathbf{u})$. The second, denoted by \mathbf{c}_r is the oriented hemispherical part of the boundary. In view of the additivity properties of the contact flux \mathbf{C} , as described in Thm.7, the resultant of I acting on \mathcal{C}_r , which is the same as the value of \mathbf{C} at the entire oriented boundary of \mathcal{C}_r , is given by

$$R_I(\mathcal{C}_r) = \mathbf{C}(\mathcal{D}_r^u, \mathbf{u}) + \mathbf{C}(\mathbf{c}_r). \quad (6.12)$$

Now let $\mathbf{n} \in \operatorname{SDc}_\Omega$ and $x \in \operatorname{Dom} \mathbf{n}$ be given and put $\mathbf{u} := \mathbf{n}(x)$. We consider the half-ball \mathcal{C}_r as described above and also a region \mathcal{G}_r which differs

from \mathcal{C}_r in that the upper disc is replaced by an oriented portion \mathbf{m}_r of the C^1 -surface $\text{Dom } \mathbf{n}$. It is clear that $\mathbf{m}_r \in \text{SDc}_\Omega$ and $\mathbf{m}_r \subset \mathbf{n}$. We denote the oriented spherical part of the boundary of \mathcal{G}_r by \mathbf{g}_r (see Fig.4).



For the region \mathcal{G}_r the equation (6.12) changes to

$$R_I(\mathcal{G}_r) = \mathbf{C}(\mathbf{m}_r) + \mathbf{C}(\mathbf{g}_r) . \quad (6.13)$$

The oriented spherical surfaces \mathbf{c}_r and \mathbf{g}_r have a portion \mathbf{h}_r in common. They are the disjoint unions of \mathbf{h}_r with portions \mathbf{c}_r^C and \mathbf{g}_r^G , respectively. In view of the additivity properties of the contact flux \mathbf{C} we have

$$\mathbf{C}(\mathbf{c}_r) = \mathbf{C}(\mathbf{h}_r) + \mathbf{C}(\mathbf{c}_r^C) , \quad \mathbf{C}(\mathbf{g}_r) = \mathbf{C}(\mathbf{h}_r) + \mathbf{C}(\mathbf{g}_r^G) \quad (6.14)$$

Subtracting (6.13) from (6.12) and using (6.14), we obtain

$$\mathbf{C}(\mathcal{D}_r^u, \mathbf{u}) - \mathbf{C}(\mathbf{m}_r) = R_I(\mathcal{C}_r) - R_I(\mathcal{G}_r) - \mathbf{C}(\mathbf{c}_r^C) + \mathbf{C}(\mathbf{g}_r^G) . \quad (6.15)$$

Since the volumes of \mathcal{G}_r and \mathcal{C}_r are proportional to r^3 it follows from (a) that

$$\lim_{r \rightarrow 0} \frac{R_I(\mathcal{G}_r)}{r^2} = 0 = \lim_{r \rightarrow 0} \frac{R_I(\mathcal{C}_r)}{r^2} . \quad (6.16)$$

Since the C^1 - surface $\text{Dom } \mathbf{n}$ is tangent to the disc \mathcal{D}_r^u , the areas of $\text{Dom } \mathbf{c}_r^C$ and $\text{Dom } \mathbf{g}_r^G$ approach 0 faster than r^2 as r approaches 0. Hence it follows from (b) that

$$\lim_{r \rightarrow 0} \frac{\mathbf{C}(\mathbf{c}_r^C)}{r^2} = 0 = \lim_{r \rightarrow 0} \frac{\mathbf{C}(\mathbf{g}_r^G)}{r^2} . \quad (6.17)$$

We also have

$$\lim_{r \rightarrow 0} \frac{\text{ar}(\text{Dom } \mathbf{m}_r)}{\text{ar}(\mathcal{D}_r^u)} = 1 = \lim_{r \rightarrow 0} \frac{\text{ar}(\text{Dom } \mathbf{m}_r)}{\pi r^2} . \quad (6.18)$$

Therefore, using (6.16), (6.17), and (6.18), we conclude from (6.15) that

$$\lim_{r \rightarrow 0} \frac{C(\mathbf{m}_r)}{\text{area}(\text{Dom } \mathbf{m}_r)} = \lim_{r \rightarrow 0} \frac{C(\mathcal{D}_r^u, \mathbf{u})}{\pi r^2}. \quad (6.19)$$

Applying (6.10) to the case when $\mathcal{T} := \mathcal{D}_r^u$, we conclude that the right side of (6.19) is $\mathbf{s}(x, \mathbf{u})$. Applying the equation (6.8) of the condition (c) to the case when $\mathbf{m} := \mathbf{m}_r$, we see that the left side of (6.19) is $\mathbf{s}_n(x)$. Since $\mathbf{n} \in \text{SDc}_\Omega$ and $x \in \text{Dom } \mathbf{n}$ were arbitrary, it follows that the function (6.1) is indeed a proto-contactor when its values are defined as in the beginning of this proof. ■

My proof of Theorem IV on p.275 in [N1] used a cylindrical region instead of a half-ball and hence was more complicated than the proof above.

Theorem B. *If I satisfies the condition (a) and admits a continuous proto-contactor \mathbf{s} relative to SDc_Ω , then I admits a continuous contactor relative to SDc_Ω .*

The traditional proof of this Theorem uses a Cartesian coordinate system. Since 1956, I have used a coordinate-free proof in my courses on Continuum Mechanics at CMU. This proof has been presented in some textbooks, for example in [T], pp.175-177. The following is an outline of a new and simpler version of this proof:

Let $x \in \mathcal{B}$, $\mathbf{u} \in \text{Usph } \mathcal{V}$ and linearly independent $\mathbf{v}, \mathbf{w} \in \{\mathbf{u}\}^\perp$ be given. Put $\mathbf{z} := \mathbf{v} + \mathbf{w}$ and let $\alpha \in \mathbb{P}^\times$ be given. Consider the triangle \mathcal{T} and the prism shown below.

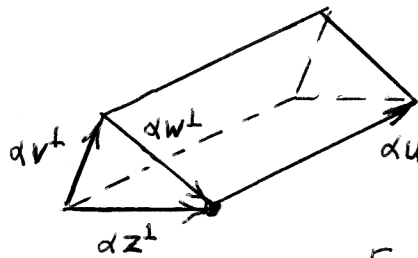
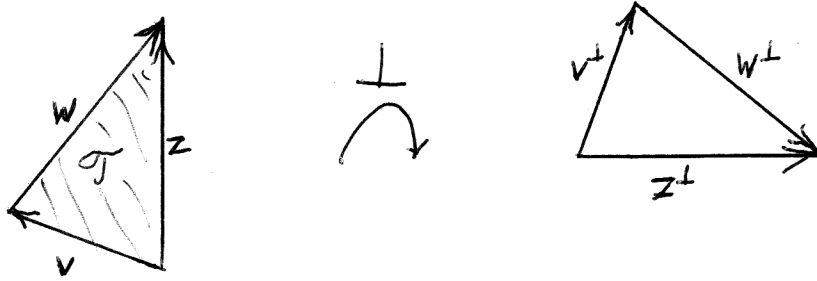


Fig. 5

Put

$$\mathbf{a} := \mathbf{s}\left(x, \frac{\mathbf{v}}{|\mathbf{v}|}\right)|\mathbf{v}| + \mathbf{s}\left(x, \frac{\mathbf{w}}{|\mathbf{w}|}\right)|\mathbf{w}| + \mathbf{s}\left(x, -\frac{\mathbf{z}}{|\mathbf{z}|}\right)|\mathbf{z}|, \quad (6.20)$$

and

$$\mathbf{b} := (\mathbf{s}(x, -\mathbf{u}) + \mathbf{s}(x, \mathbf{u}))\text{ar}(\mathcal{T}). \quad (6.21)$$

Consider the resultant R_I of I acting on the prism. Taking the limit $\alpha \rightarrow 0$, and using (a) as well as the assumed continuity of \mathbf{s} we obtain $\mathbf{a} = \mathbf{b}$.

Given $\beta \in \mathbb{P}^\times$, replace \mathbf{v} by $\beta\mathbf{v}$ and \mathbf{w} by $\beta\mathbf{w}$. Then $\mathbf{a} = \mathbf{b}$ is replaced by

$$\beta\mathbf{a} = \beta^2\mathbf{b}.$$

Since $\mathbf{a} = \mathbf{b}$ and since $\beta \in \mathbb{P}^\times$ was arbitrary, we get

$$\mathbf{a} = \mathbf{0} = \mathbf{b},$$

which shows that $\mathbf{s}(x, \cdot)$ is the restriction of a linear mapping $\mathbf{S}(x) \in \text{Lin}(\mathcal{V}, \mathcal{W})$ to the unit sphere $\text{Usph } \mathcal{V}$. ■

Remarks: Note that Theorem A only assures the existence of a proto-contactor \mathbf{s} if (a) and (c) are satisfied. Analyzing the proof, one can conclude that, for every $\mathbf{u} \in \text{Usph } \mathcal{V}$ the restrictions of $\mathbf{s}(\cdot, \mathbf{u})$ to plane regions perpendicular to \mathbf{u} are continuous. But this is much less than continuity of \mathbf{s} itself. It is not clear whether the conditions (a) and (c) alone are sufficient to insure the existence of a *continuous* proto-contactor.

If the Interaction I does admit a continuous proto-contactor and hence a contactor by Thm.B, it is clear from (6.5) that

$$\mathbf{C}(-\mathbf{n}) = -\mathbf{C}(\mathbf{n}) \quad \text{for all } \mathbf{n} \in \text{SDC}_\Omega. \quad (6.22)$$

In view of Prop.3, it follows that I is C^1 -skew in the sense that (2.5) holds for all $(\mathcal{P}, \mathcal{Q}) \in (\Omega^2)_{\text{sep}}$ such that $\text{Rct}(\mathcal{P}, \mathcal{Q})$ is a C^1 -surface.

If the conditions (a) and (c) are satisfied but if the resulting proto-contactor of Thm.A fails to be continuous, it is not clear whether (6.22) remains valid. This question can only be answered by a proof or a counter-example.

It is also not clear what happens if the Class SDC_Ω is replaced by a larger subclass of DC_Ω or DC_Ω itself. However, investigating this issue is likely to involve sophisticated arguments from geometric measure theory.

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