Department of Mathematical Sciences
Carnegie Mellon University

21-476 **Ordinary Differential Equations** **Fall 2003**

X. Linear System: Basic Concepts

Let n be a positive integer and $||\cdot||$ be a norm on $\mathbb{R}^n$. We denote by $\mathbb{R}^{n\times n}$ the set of all $n\times n$ matrices with real entries. With the usual notions of matrix addition and multiplication of a matrix by a scalar, $\mathbb{R}^{n\times n}$ is a real linear space of dimension n^2. We define a norm $|||\cdot|||$ on $\mathbb{R}^{n\times n}$ by

$$|||B||| = \max\ \{||Bz|| : z\in\mathbb{R}^n,\ ||z||=1\} \qquad \text{for all } B\in\mathbb{R}^{n\times n}. \tag{10.1}$$

[The elements of $\mathbb{R}^n$ are to be treated as column vectors in situations involving matrix multiplication.] It follows easily from (10.1) that

$$|||Bz|| \le |||B|||\cdot||z|| \qquad \text{for all } B\in\mathbb{R}^{n\times n},\ z\in\mathbb{R}^n. \tag{10.2}$$

Observe that different norms on $\mathbb{R}^n$ lead to different norms on $\mathbb{R}^{n\times n}$.

Let $I\subset\mathbb{R}$ be a nonempty open interval and let $A: I\to\mathbb{R}^{n\times n}$ and $g: I\to\mathbb{R}^n$ be given. We assume throughout this section that A and g are continuous on I. Consider the linear system of differential equations

$$\dot{x}(t) = A(t)x(t) + g(t). \tag{L}$$

If we put $D = I\times\mathbb{R}^n$ and define $f: D\to\mathbb{R}^n$ by

$$f(t,z) = A(t)z + g(t) \qquad \text{for all } (t,z)\in D,$$

it is easy to see that f is continuous and locally Lipschitzean on D. Consequently, for each $t_0\in I,\ x_0\in\mathbb{R}^n$ we have existence of exactly one noncontinuable solution x of (L) satisfying $x(t_0)=x_0$. Moreover, it is straightforward to show that every noncontinuable solution x of (L) has $\mathrm{Dom}(x) = I$. Therefore, when we speak of a solution of (L) we shall mean a solution with domain I, unless stated otherwise.

Observe that if x and x^* are solutions of (L) then $x - x^*$ is a solution of the corresponding homogeneous equation

$$\dot{x}(t) = A(t)x(t). \tag{LH}$$

Since multiplication by an element of $\mathbb{R}^{n\times n}$ maps $\mathbb{R}^{n\times n}$ into $\mathbb{R}^{n\times n}$, it is possible to consider matrix-valued solutions of (LH). By a matrix-valued solution of (LH) we mean a differentiable function $X: I\to\mathbb{R}^{n\times n}$ such that $\dot{X}(t) = A(t)X(t)$ for all $t\in I$. Note that a matrix-valued function $X: I\to\mathbb{R}^{n\times n}$ is a solution of (LH) if and only if each column is a solution of (LH).

Proposition 10.1:

(i) If x and x^* are solutions of (L) then $x - x^*$ is a solution of (LH).

(ii) The set of all solutions of (LH) is a (real) linear space.

(iii) If x is a solution of (L) and y is a solution of (LH) then $x + y$ is a solution of (L).

Proposition 10.2: Let X be a matrix-valued solution of (LH) and let $\xi \in \mathbb{R}^n$, $C \in \mathbb{R}^{n\times n}$ be given. Define $x : I \to \mathbb{R}^n$ by $x(t) = X(t)\xi$ and $Y : I \to \mathbb{R}^{n\times n}$ by $Y(t) = X(t)C$ for all $t \in I$. Then x is a solution of (LH) and Y is a matrix-valued solution of (LH).

Proposition 10.3: Let X be a matrix-valued solution of (LH). Then either $X(t)$ is invertible for all $t \in I$ or $X(t)$ is singular for all $t \in I$.

Definition 10.4: A matrix-valued solution X of (LH) is called a fundamental matrix solution if $X(t)$ is invertible for all $t \in I$.

Proposition 10.5: (LH) has a fundamental matrix solution. (In fact, there are many.)

Proposition 10.6: Let X be a fundamental matrix solution of (LH) and define $Y : I \to \mathbb{R}^{n\times n}$ by $Y(t) = (X(t))^{-1}$ for all $t \in I$. Then Y satisfies $\dot{Y}(t) = -Y(t)A(t)$ for all $t \in I$.

Proposition 10.7: (Abel's Equation): Let X be a matrix-valued solution of (LH) and define $\varphi(t) = \det(X(t))$ for all $t \in I$. Then φ satisfies

$$\dot{\varphi} = [trA(t)]\varphi(t) \qquad \text{for all } t \in I,$$

where $trA(t)$ denotes the trace of $A(t)$, i.e. the sum of the entries on the main diagonal.

Proposition 10.8 (Variation of Constants) Let X be a fundamental matrix solution of (LH) and let $t_0 \in I,\ x_0 \in \mathbb{R}^n$ be given. The solution x of (L) satisfying $x(t_0) = x_0$ is given by

$$x(t) = X(t)X(t_0)^{-1}x_0 + \int_{t_0}^{t} X(t)X(s)^{-1}g(s)ds.$$