Department of Mathematical Sciences
Carnegie Mellon University

21-476 **Ordinary Differential Equations** **Fall 2003**

XI. Matrix Exponential

In this section we consider homogeneous linear systems with constant coefficients, i.e. autonomous versions of (LH). It is convenient to allow the coefficients and solutions to be complex-valued.

Indeed, complex-valued solutions are often helpful for constructing real-valued solutions to systems with real coefficients. Let $A \in \mathbb{C}^{n\times n}$ be given and consider the system

$$\text{(ALH)} \qquad \dot{x}(t) = Ax(t).$$

By a solution of (ALH) we mean a differentiable function $x : \mathbb{R} \to \mathbb{C}^n$ such that (ALH) holds for all $t \in \mathbb{R}$. By a matrix-valued solution of (ALH) we mean a differentiable function $X : \mathbb{R} \to \mathbb{C}^{n\times n}$ such that $\dot{X}(t) = AX(t)$ for all $t \in \mathbb{R}$. Notice that a function $X : \mathbb{R} \to \mathbb{C}^{n\times n}$ is a matrix-valued solution of (ALH) if and only if each column is a solution of (ALH). Notice also that if X is a matrix-valued solution of (ALH) and $\xi \in \mathbb{C}^n$, $C \in \mathbb{C}^{n\times n}$ then $t \to X(t)\xi$ is a solution of (ALH) and $t \to X(t)C$ is a matrix-valued solution of (ALH). It is straightforward to verify that a matrix -valued solution of (ALH) is invertible for all times if and only if it is invertible at 0.

Definition 11.1 For each $t \in \mathbb{R}$ we define $e^{tA} \in \mathbb{C}^{n\times n}$ to be the value at t of the matrix-valued solution X of (ALH) satisfying $X(0) = I$, where I is the $n \times n$ identify matrix.

Proposition 11.2 Let $A, B \in \mathbb{C}^{n\times n}$ be given. Then

(i) $e^{0A} = I$;

(ii) $e^{(t+s)A} = e^{tA}e^{sA}$ for all $s, t \in \mathbb{R}$;

(iii) $\left(e^{tA}\right)^{-1} = e^{-tA}$ for all $t \in \mathbb{R}$;

(iv) $Ae^{tA} = e^{tA}A$ for all $t \in \mathbb{R}$;

(v) $e^{tA} = \sum_{m=0}^{\infty} \frac{(tA)^m}{m!}$ for all $t \in \mathbb{R}$;

(vi) If B is invertible then $B^{-1}e^{tA}B = e^{tB^{-1}AB}$ for all $t \in \mathbb{R}$;

(vii) If $A = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$ then $e^{tA} = \text{diag}\,(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})$ for all $t \in \mathbb{R}$;

(viii) $\det\left(e^{tA}\right) = \exp[tr(A)t]$ for all $t \in \mathbb{R}$;

(ix) $Be^{tA} = e^{tA}B$ for all $t \in \mathbb{R}$ if and only if $AB = BA$;

(x) $e^{t(A+B)} = e^{tA}e^{tB}$ for all $t \in \mathbb{R}$ if and only if $AB = BA$.