Department of Mathematical Sciences
Carnegie Mellon University

21-476 **Ordinary Differential Equations** **Fall 2003**

XIII. Stability and Linearization

Let $A \in \mathbb{R}^{n\times n}$ be given and consider the system

$$\dot{x} = Ax. \tag{ALH}$$

The stability of the zero solution of (ALH) can be characterized completely by studying the eigenvalues and eigenvectors of A. Here it should be emphasized that even though A is a real matrix when we speak of eigenvalues and eigenvectors of A we mean eigenvalues and eigenvectors of A regarded as a linear mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$.

Theorem 13.1:

(i) The zero solution of (ALH) is stable if and only if Re $\lambda \leq 0$ for all $\lambda \in \sigma(A)$ and for every $\lambda \in \sigma(A)$ with Re $\lambda = 0$ we have dim $\mathcal{N}(\lambda I - A) = m_A(\lambda)$.

(ii) The zero solution of (ALH) is asymptotically stable if and only if Re $\lambda < 0$ for all $\lambda \in \sigma(A)$.

Corollary 13.2: If $tr(A) > 0$ then the zero solution of (ALH) is unstable.

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be given and consider the autonomous system

$$\dot{x} = g(x). \tag{A}$$

Suppose that x^* is a critical point for (A) and that g is continuously differentiable. Then for all $z \in \mathbb{R}^n$ with $||z||$ sufficiently small we have

$$g(x^* + z) \approx \nabla g(x^*)z. \tag{13.1}$$

If we put $y = x - x^*$, then (A) becomes

$$\dot{y} = g(x^* + y). \tag{13.2}$$

In view of (13.1) we expect that (13.2) can be approximated by

$$\dot{y} = \nabla g(x^*)y \tag{13.3}$$

as long as $||y||$ is sufficiently small. It is therefore reasonable to expect that the stability of x^* as a critical point of (A) is related to the stability of the zero solution of (13.3). This would be very useful because the stability of the zero solution of (13.3) has been completely characterized in Theorem 13.1. Equation (13.3) is called the linearization of (A) about x^*. Except for cases when each eigenvalue of $\nabla g(x^*)$ has real part ≤ 0 and at least one eigenvalue has real part $= 0$, the stability of x^* as a critical point of (A) is the same as the stability of the zero solution of (13.3).

Theorem 13.3: Assume that g is continuously differentiable and that $g(x^*) = 0$.

(i) If $\text{Re } \lambda < 0$ for all $\lambda \in \sigma(\nabla g(x^*))$ then x^* is an asymptotically stable critical point of (A).

(ii) If there exists $\lambda \in \sigma(\nabla g(x^*))$ with $\text{Re } \lambda > 0$ then x^* is an unstable critical point of (A). In particular, if $tr(\nabla g(x^*)) > 0$ then x^* is unstable.