Department of Mathematical Sciences
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21-476 **Ordinary Differential Equations** **Fall 2003**

XII. Some Remarks on Eigenvectors and Generalized Eigenvectors

Let $A \in \mathbb{C}^{n\times n}$ be given. A complex number λ is called an *eigenvalue* of A if the null space of $\lambda I - A$ is nontrivial, i.e. if $\mathcal{N}(\lambda I - A) \neq \{0\}$. Here I is the $n\times n$ identity matrix, and for each $B \in \mathbb{C}^{n\times n}$, $\mathcal{N}(B) = \{\xi \in \mathbb{C}^n : B\xi = 0\}$. If λ is an eigenvalue of A the nonzero elements of $\mathcal{N}(\lambda I - A)$ are called *eigenvectors* associated with λ. The set of all eigenvalues of A is called the *spectrum* of A and is denoted by $\sigma(A)$. The eigenvalues of A are precisely the roots of the *characteristic equation*.

$$P_A(\lambda) = 0, \tag{12.1}$$

where $P_A : \mathbb{C} \to \mathbb{C}$ is the *characteristic polynomial* and is defined by

$$P_A(\lambda) = \det(\lambda I - A) \qquad \text{for all } \lambda \in \mathbb{C}. \tag{12.2}$$

P_A is a polynomial of degree n and consequently $\sigma(A)$ is nonempty and contains at most n elements. The *algebraic multiplicity* of an eigenvalue λ of A is defined to be its multiplicity as a root of (12.1) and is denoted by $m_A(\lambda)$.

Proposition 12.1:

(i) $tr(A) = \sum_{\lambda\in\sigma(A)} m_A(\lambda)\lambda$

(ii) $\det(A) = \prod_{\lambda\in\sigma(A)} \lambda^{m_A(\lambda)}$

Notice that if λ is an eigenvalue of A and ξ is an associated eigenvector then $e^{tA}\xi = e^{\lambda t}\xi$ for all $t \in \mathbb{R}$. Consequently, if A has n linearly independent eigenvectors then we have a simple representation for e^{tA}.

Proposition 12.2: Assume that $\sigma(A)$ contains exactly n elements (i.e. that $m_A(\lambda) = 1$ for every $\lambda \in \sigma(A)$). Then $\dim(\lambda I - A) = 1$ for every $\lambda \in \sigma(A)$ and there is a basis for $\mathbb{C}^n$ consisting solely of eigenvectors of A.

If $\sigma(A)$ contains strictly less than n elements there may or may not be n linearly independent eigenvectors. However, there is always a basis that can be used to obtain a convenient representation for e^{tA}.

Definition 12.3: Let λ be an eigenvalue of A. A nonzero vector $\xi \in \mathbb{C}^n$ is called a *generalized eigenvector* associated with λ if there is a positive integer k such that $\xi \in \mathcal{N}\left((\lambda I - A)^k\right)$.

Remark 12.3: Let λ be an eigenvalue of A and ξ be an associated generalized eigenvector, and choose a positive integer k such that $(\lambda I - A)^k \xi = 0$. Notice that $(\lambda I - A)^m \xi = 0$ for all integers $m \geq k$. Therefore, we have

$$\begin{aligned} e^{tA}\xi &= e^{t\lambda I} e^{t(A-\lambda I)}\xi = e^{\lambda t} e^{t(A-\lambda I)}\xi \\ &= e^{\lambda t}\left(\sum_{m=0}^{\infty} \frac{t^m}{m!}(A-\lambda I)^m\right)\xi \\ &= e^{\lambda t}\left(\xi + t(A-\lambda I)\xi + \ldots + \frac{t^{k-1}}{(k-1)!}(A-\lambda I)^{k-1}\xi\right) \end{aligned}$$

Theorem 12.4: For each $\lambda \in \sigma(A)$ there is exactly one integer $r_A(\lambda)$ satisfying

(i) $1 \leq r_A(\lambda) \leq m_A(\lambda)$

(ii) $\dim \mathcal{N}\left((\lambda I - A)^{r_A(\lambda)}\right) = m_A(\lambda)$

(iii) $\mathcal{N}\left((\lambda I - A)^m\right) = \mathcal{N}\left((\lambda I - A)^{r_A(\lambda)}\right)$ for all $m \in \mathbb{N}$ with $m \geq r_A(\lambda)$

(iv) $\mathcal{N}\left((\lambda I - A)^{r_A(\lambda)-1}\right) \neq \mathcal{N}\left((\lambda I - A)^{r_A(\lambda)}\right)$

Theorem 12.5: There is a basis $\mathcal{B}$ for $\mathbb{C}^n$ with the following properties.

(i) Every element of $\mathcal{B}$ is a generalized eigenvector of A.

(ii) For every $\lambda \in \sigma(A)$ there are exactly $m_A(\lambda)$ elements of $\mathcal{B}$ that belong to $\mathcal{N}\left((\lambda I - A)^{r_A(\lambda)}\right)$