Department of Mathematical Sciences
Carnegie Mellon University

21-476 **Ordinary Differential Equations** **Fall 2003**

VIII. Autonomous Systems: Stability and Liapunov Function

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be given and consider the autonomous system

$$\dot{x} = g(x). \tag{A}$$

We assume throughout this section that g is continuous and has the uniqueness property. For each $y \in \mathbb{R}^n$, $\delta > 0$ we put

$$B_\delta(y) = \{z \in \mathbb{R}^n : ||z - y|| < \delta\}. \tag{8.1}$$

Definition 8.1: Let x^* be a critical point for (A). We say that x^* is

(a) stable if $\forall \epsilon > 0$, $\exists \delta > 0$ such that for every $p \in B_\delta(x^*)$ we have $\eta_+(p) = \infty$ and $||\varphi(t, p) - x^*|| < \epsilon$ for all $t \geq 0$.

(b) asymptotically stable if it is stable and $\exists \delta' > 0$ such that for every $p \in B_{\delta'}(x^*)$ we have $\eta_+(p) = \infty$ and $\varphi(t, p) \to x^*$ as $t \to \infty$.

(c) unstable if it is not stable.

Theorem 8.2 (Liapunov): Let x^* be a critical point for (A) and U be an open subset of $\mathbb{R}^n$ with $x^* \in U$. Assume that $V : U \to \mathbb{R}$ is continuously differentiable and define $\dot{V} : U \to \mathbb{R}$ by

$$\dot{V}(z) = \nabla V(z) \cdot g(z) \qquad \forall z \in U.$$

(a) If V is locally positive definite at x^* and $\dot{V}$ is locally negative semidefinite at x^* then x^* is stable.

(b) If V is locally positive definite at x^* and $\dot{V}$ is locally negative definite at x^* then x^* is asymptotically stable.

(c) If $V(x^*) = 0$, $\dot{V}$ is locally positive definite at x^* and $\forall \delta > 0$, $\exists z \in B_\delta(x^*) \cap U$ with $V(z) > 0$ then x^* is unstable.