21-341 **Linear Algebra I** **Fall 2003**

Review Problems for Test 3

Let $\mathbb{F}$ be a field and V be a finite-dimensional vector space over $\mathbb{F}$.

Given $m, n, \in \mathbb{Z}^+$, $\mathbb{F}^{m\times n}$ denotes the set of all $m \times n$ matrices with entries from $\mathbb{F}$.

1. Let $\mathbb{F} = \mathbb{R}$ and

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 4 & 7 \\ 2 & 4 & 9 & 9 \\ -1 & -2 & -3 & -3 \end{pmatrix}$$

   Find $\det(A)$.

2. Let $\mathbb{F} = \mathbb{R}$ and assume that $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is an inner product. Let $T \in L(V,V)$ be given and assume that $(Tx, y) = -(x, Ty) \quad \forall x, y \in V$. Let U be a subspace of V and let

$$U^\perp = \{x \in V : (x, y) = 0 \quad \forall y \in U\}.$$

   Show that if U is T-invariant then $U^\perp$ is T-invariant.

3. Let $A \in \mathbb{F}^{9\times 8}$ and $B \in \mathbb{F}^{8\times 9}$ be given and let $C = AB$. Show that $\det(C) = 0$.

4. Let $T \in L(V,V)$ be given and assume that $T^3 = T$. What are the possible values of $\det(T)$? What are the possible eigenvalues for T?

5. Let $T \in L(V,V)$ and $k \in \mathbb{Z}^+$ be given. Assume that $T^k = 0$ (Such a linear transformation is called nilpotent.) Show that 0 is an eigenvalue for T and that T has no other eigenvalues.

6. Let $\mathbb{F} = \mathbb{Z}_5$ and let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$

   Compute $\det(A)$.

7. Assume that $\dim V$ is odd and that $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is an inner product. Let $T \in L(V,V)$ be given and assume that $(Tx, y) = -(x, Ty)$ for all $x, y \in V$. Show that $\det(T) = 0$.

8. Let $\mathbb{F} = \mathbb{R}$ and let

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

   (a) Find the minimal polynomial for A.
   (b) Find the characteristic polynomial for A.
   (c) Find all eigenvalues and eigenvectors for A.
   (d) Is A diagonalizable? Explain.

9. Let $\mathbb{F} = \mathbb{R}$ and let

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}.$$

   Find all eigenvalues and eigenvectors for A.

10. Let $\mathbb{F} = \mathbb{R}$ and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}.$$

   Find all eigenvalues and eigenvectors for A.

11. Let $\mathbb{F} = \mathbb{C}$ and

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

   (a) Find all eigenvalues and eigenvectors for A.
   (b) Is A diagonalizable? If so, find $S \in \mathbb{C}^{3\times 3}$ such that $S^{-1}AS$ is diagonal.

12. Let $T \in L(V, V)$ and $\lambda \in \mathbb{F}\backslash\{0\}$ be given. Assume that T is invertible. Show that λ is an eigenvalue for T if and only if λ^{-1} is an eigenvalue for T^{-1}.

13. Let $n \in \mathbb{Z}$ and $A \in \mathbb{F}^{n\times n}$ be given. Assume that A has n distinct eigenvalues $\lambda_1, \lambda_2 \ldots, \lambda_n$. Show that

$$\det(A) = \prod_{i=1}^{n} \lambda_i, \quad \text{and} \quad tr(A) = \sum_{i=1}^{n} \lambda_i.$$