Department of Mathematical Sciences
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21-476 **Ordinary Differential Equations** **Fall 2003**

V. Periodic Systems

Let $T > 0$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be given. Throughout this section we assume that f is continuous, has the uniqueness property, and satisfies

$$f(t+T, z) = f(t, z) \qquad \forall t \in \mathbb{R},\ z \in \mathbb{R}^n. \tag{5.1}$$

By a *T-periodic solution* of

$$\dot{x}(t) = f(t, x(t)) \tag{DE}$$

we mean a solution x of (DE) such that $\mathrm{Dom}(x) = \mathbb{R}$ and

$$x(t+T) = x(t) \qquad \forall t \in \mathbb{R}. \tag{5.2}$$

The following lemma is a direct consequence of the uniqueness property and (5.1).

Lemma 5.1: *Let x be a noncontinuable solution of (DE) and let $t_0 \in \mathrm{Dom}(x)$ be given. If $t_0 + T \in \mathrm{Dom}(x)$ and $x(t_0 + T) = x(t_0)$ then x is a T-periodic solution.*

By using Lemma 5.1, together with Theorem 4.11 and Brouwer's fixed point Theorem, we obtain the following important result.

Theorem 5.2: *Let S be a nonempty, closed, bounded, convex subset of $\mathbb{R}^n$ and let $t_0 \in \mathbb{R}$ be given. Assume that for every $x_0 \in S$ the unique noncontinuable solution x of*

$$\dot{x}(t) = f(t, x(t))\ ;\ x(t_0) = x_0 \tag{IVP}$$

satisfies $t_0 + T \in \mathrm{Dom}(x)$ and $x(t_0 + T) \in S$. Then (DE) has a T-periodic solution.

In order to apply Theorem 5.2 in practice, the key step is to find a suitable set S. The following lemma, which is a consequence of the Mean Value Theorem, is often helpful for this purpose.

Lemma 5.3: *Let $I \subset \mathbb{R}$ be an open interval and let $t_0 \in I$ and $\alpha,\ \alpha',\ \beta,\ \beta' \in \mathbb{R}$ with $\alpha' < \alpha$ and $\beta < \beta'$ be given.*

(a) If $F(t_0) \geq \alpha$ and $\dot{F}(s) \geq 0$ for all $s \in I \cap [t_0, \infty)$ such that $\alpha' \leq F(s) \leq \alpha$ then $F(t) \geq \alpha$ for all $t \in I \cap [t_0, \infty)$.

(b) If $F(t_0) \leq \beta$ and $\dot{F}(s) \leq 0$ for all $s \in I \cap [t_0, \infty)$ such that $\beta \leq F(s) \leq \beta'$ then $F(t) \leq \beta$ for all $t \in I \cap [t_0, \infty)$.

Theorem 5.4: *Let $\Gamma_1 \geq 0$ and $\Gamma_2 > 0$ be given. Assume that*

$$z \cdot f(t,z) \leq 0 \qquad \text{for all } t \in \mathbb{R},\ z \in \mathbb{R}^n \text{ with } \Gamma_1 \leq ||z||_2 \leq \Gamma_2. \tag{5.3}$$

Then (DE) has a T-periodic solution.