Linear Transformations and Matrices Updated 11/20/03

Let $n \in \mathbb{Z}^+$ be given. Let $\mathbb{F}$ be a field and V be a vector space over $\mathbb{F}$ with $\dim V = n$. Let $v_1, v_2, \ldots, v_n$ be a basis for V. We denote by $\mathbb{F}^{n\times 1}$ the set of all $n \times 1$ matrices with entries from $\mathbb{F}$. Consider the linear mapping $C : V \to \mathbb{F}^{n\times 1}$ defined by

$$Cv_i = e_i^t \qquad i = 1, 2, \ldots, n \tag{1}$$

where e_i^t is the $n \times 1$ matrix whose ith entry (row) is 1 and all other entries are 0. Notice that if $v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n$ then

$$Cv = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}. \tag{2}$$

We call C the component mapping for the basis $v_1, v_2, \ldots, v_n$.

I. Matrix for $T \in L(V, V)$:

Let $T \in L(V, V)$ be given. We want to find an $n \times n$ matrix A such that

$$C(Tv) = A(Cv) \quad \text{for all } v \in V. \tag{3}$$

Notice that if we have such a matrix, then we can compute Tv for a given $v \in V$ as follows: Choose $\lambda_1, \lambda_2, \ldots \lambda_n \in \mathbb{F}$ such that

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n \tag{4}$$

and let

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}. \tag{5}$$

Then we have

$$Tv = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n. \tag{6}$$

Let $x \in \mathbb{F}^{n\times 1}$ be given. If we put $v = C^{-1}x$ in (3) we obtain

$$Ax = C(TC^{-1}x) \tag{7}$$

In particular, we have

$$\begin{aligned} Ae_j^t &= C(TC^{-1}e_j^t) \\ &= C(Tv_j) \quad j = 1, 2, \ldots, n. \end{aligned} \tag{8}$$

Notice that Ae_j^t is simply the jth column of A. Therefore, the jth column of A simply consists of the coefficients needed to express Tv_j as a linear combination of $v_1, v_2, \ldots v_n$.

Change of Basis for $T \in L(V, V)$:

Let $T \in L(V, V)$ be given and let A be the matrix for T relative to the basis $v_1, v_2, \ldots, v_n$. Let $w_1, w_2, \ldots, w_n$ be a second basis for V satisfying

$$w_i = \sum_{j=1}^{n} \mu_{ji} v_j, \; i = 1, 2, \ldots, n. \tag{9}$$

Let B be the matrix for T relative to $w_1, w_2, \ldots, w_n$. We want to find the relationship between A and B.

Let S be the $n \times n$ matrix whose ij entry is μ_{ij}. Then, by (9), S is the matrix relative to $v_1, v_2, \ldots v_n$ of the linear transformation $\mathcal{S} \in L(V, V)$ characterized by

$$\mathcal{S}v_i = w_i \quad , \quad i = 1, 2, \ldots, n. \tag{10}$$

Let D be the component mapping for $w_1, w_2, \ldots w_n$ and observe that

$$Bx = DTD^{-1}x \quad \text{for all } x \in \mathbb{F}^{n\times 1} \tag{11}$$

by virtue of Part I. It follows from (10) that

$$DSv = Cv \quad \text{for all } v \in V \tag{12}$$

which yields

$$\mathcal{S}v = D^{-1}Cv \quad \text{for all } v \in V. \tag{13}$$

Using the results of Part I, we find that

$$\begin{aligned} Sx &= CD^{-1}CC^{-1}x \\ &= CD^{-1}x. \end{aligned} \quad \text{for all } x \in \mathbb{F}^{n\times 1} \tag{14}$$

Moreover, by (3), we have

$$Tv = C^{-1}ACv \quad \forall v \in V. \tag{15}$$

Substitution of (15) into (11) yields

$$\begin{aligned} Bx &= D(C^{-1}AC)D^{-1}x \\ &= S^{-1}ASx \end{aligned} \tag{16}$$

by virtue of (14).

III. Matrix for $T \in L(V, W)$:

Let $m \in \mathbb{Z}^+$ be given and let W be a vector space over $\mathbb{F}$ with $\dim W = m$. Let $u_1, u_2, \ldots u_m$ be a basis for W and let $E \in L(W, \mathbb{F}^{m\times 1})$ be the component mapping for $u_1, u_2, \ldots, u_m$.

Let $T \in L(V, W)$ be given. We want to find an $m \times n$ matrix $A \in \mathbb{F}^{m\times n}$ such that

$$ETv = ACv \quad \forall v \in V. \tag{17}$$

Let $x \in \mathbb{F}^{n\times 1}$ be given. If we put $v = C^{-1}x$ in (17) we get

$$ETC^{-1}x = Ax \quad \forall x \in \mathbb{F}^{n\times 1}. \tag{18}$$

To understand what the matrix A looks like, we set $x = e_j^t$ in (18) to get

$$\begin{array}{lcl} ETC^{-1}e_j^t & = & Ae_j^t \\ ETv_j & = & Ae_j^t \end{array} \tag{19}$$

which says that the jth column of A consists of the coefficients required to express Tv_j as a linear combination of $u_1, u_2, \ldots u_m$. We call A the matrix for T relative to the bases $v_1, v_2, \ldots v_n$ and $u_1, u_2, \ldots u_m$.

IV. Inner Product Spaces:

Suppose that $\mathbb{F} = \mathbb{R}$, $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is an inner product, and that $u_1, u_2, \ldots u_n$ is an orthormal basis for V. Since

$$v = \sum_{i=1}^{n} (v, u_i) u_i \quad \forall v \in V, \tag{20}$$

it follows that

$$Cv = \begin{pmatrix} (v, u_1) \\ (v, u_2) \\ \vdots \\ (v, u_n) \end{pmatrix} \tag{21}$$

Let $T \in L(V, V)$ be given. It follows from (8) and (21) that if A is the matrix for T relative to $u_1, u_2, \ldots u_n$ then

$$A_{ij} = (Tu_j, u_i), \tag{22}$$

where A_{ij} is the entry of A from row i and column j.