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21-476 **Ordinary Differential Equations** **Fall 2003**

IX. Poincáre-Bendixson Theory for Planar Autonomous System

For autonomous systems with $n = 2$ there is a very rich and elegant theory concerning periodic orbits. The basis for this theory is the Jordan Curve Theorem. A *Jordan curve* in $\mathbb{R}^2$ is a set $J \subset \mathbb{R}^2$ that is homeomorphic to $\{z \in \mathbb{R}^2 : z_1^2 + z_2^2 = 1\}$. A set $S \subset \mathbb{R}^2$ is said to be *arcwise connected* if for every $p, q \in S$ there is a continuous function $F : [0,1] \to \mathbb{R}^2$ such that $F(0) = p$, $F(1) = q$ and $F(t) \in S$ for all $t \in [0,1]$.

Theorem 9.1 (Jordan Curve Theorem): Let J be a Jordan curve in $\mathbb{R}^2$. Then there exist unique, nonempty, open sets $U_i, U_e \subset \mathbb{R}^2$ such that $U_i \cap U_e = \emptyset$, U_i is bounded and $\mathbb{R}^2 \backslash J = U_i \cup U_e$. The sets U_i and U_e are arcwise connected; U_i is called the *region interior to* J and U_e is called the *region exterior to* J.

Although it may seem almost self-evident, the Jordan Curve Theorem is a deep result in Topology.

Theorem 9.2 (Schoenflies Theorem): Let J be a Jordan curve in $\mathbb{R}^2$ and U_i be the region interior to J. Then $J \cup U_i$ is homeomorphic to $\{z \in \mathbb{R}^2 : ||z||_2 \leq 1\}$.

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be given and consider the autonomous system

$$\dot{x} = g(x). \tag{9.1}$$

Throughout this section we assume that g is continuous and satisfies the uniqueness property.

Lemma 9.3: Let J be a Jordan curve and U_i, U_e be the regions interior to J and exterior to J, respectively. Let $p \in \mathbb{R}^2$ be given. If $\gamma^+(p) \cap U_i \neq \emptyset$ and $\gamma^+(p) \cap U_e \neq \emptyset$, then $\gamma^+(p) \cap J \neq \emptyset$.

Lemma 9.4: Let $p \in \mathbb{R}^2$ be given and assume that $\gamma(p)$ is a periodic orbit. Then $\gamma(p)$ is a Jordan curve.

Theorem 9.5: Let $p \in \mathbb{R}^2$ be given and assume that $\gamma(p)$ is a periodic orbit. Then the region interior to $\gamma(p)$ must contain a critical point.

Theorem 9.6: Assume that g is continuously differentiable. Let $p \in \mathbb{R}^2$ be given and assume that $\gamma(p)$ is a periodic orbit. Then

$$\iint_{U_i} \operatorname{div} g \, dA = 0,$$

where U_i denotes the region interior to $\gamma(p)$.

Definition 9.7: Let S be an open subset of $\mathbb{R}^2$. We say that S is simply connected if it is arcwise connected and for every Jordan curve $J \subset S$ we have $U_i \subset S$, where U_i is the region interior to J.

Corollary 9.8: (Negativity Criterion of Bendixson): Assume that g is continuously differentiable and let S be a simply connected open subset of $\mathbb{R}^2$. Assume further that for every nonempty bounded open set $D \subset S$ we have

$$\iint_{D} \operatorname{div} g \, dA \neq 0.$$

Then there are no periodic orbits that lie entirely in S.

Theorem 9.9: Let $p \in \mathbb{R}^2$ be given and assume that $\gamma^+(p)$ is bounded. If $\gamma^+(p) \cap \omega(p) \neq \emptyset$ then either p is a critical point or $\gamma(p)$ is a periodic orbit.

Theorem 9.10: (Poincaré-Bendixson) Let $p \in \mathbb{R}^2$ be given and assume that $\gamma^+(p)$ is bounded. If $\omega(p)$ contains no critical points then $\omega(p)$ is a periodic orbit.

Corollary 9.11: Let S be a nonempty, closed, bounded subset of $\mathbb{R}^2$. If S is positively invariant and contains no critical points, then S contains a periodic orbit.