Department of Mathematical Sciences
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21-476 **Ordinary Differential Equations** **Fall 2003**

I. Review of Some Solution Techniques for Single First-Order Equations

1. **Linear Equations**: Let I be an interval and assume that $p, q : I \to \mathbb{R}$ are continuous. Given $t_0 \in I$ and $x_0 \in \mathbb{R}$, consider the initial-value problem

$$\dot{x}(t) + p(t)x(t) = q(t), \quad x(t_0) = x_0. \tag{1.1}$$

To solve (1.1) we chose $P : I \to \mathbb{R}$ such that $\dot{P}(t) = p(t)$ for all $t \in I$ and put $\mu(t) = \exp(P(t))$ for all $t \in I$. (Such a function μ is called an *integrating factor*.) Observe that

$$\dot{\mu}(t) = \exp(P(t))\dot{P}(t) = \mu(t)p(t). \tag{1.2}$$

Multiplying the differential equation by μ and making use of (1.2) we find that

$$\mu(t)\dot{x}(t) + \dot{\mu}(t)x(t) = \mu(t)q(t) \tag{1.3}$$

or

$$\frac{d}{dt}(\mu(t)x(t)) = \mu(t)q(t), \tag{1.4}$$

which can be integrated to find x. The solution of (1.1) is given by

$$x(t) = \frac{1}{\mu(t)}\left[\mu(t_0)x_0 + \int_{t_0}^{t} \mu(s)q(s)ds\right]. \tag{1.5}$$

2. **Separation of Variables**: Let I and J be open intervals and assume that $g : I \to \mathbb{R}$ is continuous and $h : J \to \mathbb{R}$ is continuously differentiable. Given $t_0 \in I$ and $x_0 \in J$, consider the initial value problem

$$\dot{x}(t) = g(t)h(x(t));\ x(t_0) = x_0. \tag{1.6}$$

It can be shown that either the solution is constant, i.e. $x(t) = x_0$ for all $t \in I$ or $h(x(t))$ never vanishes. It is easy to check whether or not the constant function $x(t) = x_0$ satisfies the differential equation. Suppose the solution of (1.6) is nonconstant. Then $h(x(t))$ never vanishes and we may rewrite the differential equation as

$$\frac{1}{h(x(t))}\dot{x}(t) = g(t). \tag{1.7}$$

Let J_0 be the largest interval such that $x_0 \in J_0 \subset J$ and h does not vanish on J_0. We choose $H : J_0 \to \mathbb{R}$ such that

$$H'(z) = \frac{1}{h(z)} \qquad \text{for all } z \in J_0. \tag{1.8}$$

Then we may rewrite (1.7) as

$$\frac{d}{dt}H(x(t)) = g(t), \tag{1.9}$$

which can be integrated to obtain

$$H(x(t)) = H(x_0) + \int_{t_0}^{t} g(s)ds. \tag{1.10}$$

3. **Exact Equations**: Let D be a simply connected* open subset of $\mathbb{R}^2$ and assume that $M, N : D \to \mathbb{R}$ are continuously differentiable. Consider the differential equation

$$M(t, x(t)) + N(t, x(t))\dot{x}(t) = 0. \tag{1.11}$$

Equation (1.11) is said to be *exact* if there exists a function $\psi : D \to \mathbb{R}$ such that

$$\psi_{,1} = M \text{ and } \psi_{,2} = N \text{ on } D, \tag{1.12}$$

where $\psi_{,1}$ and $\psi_{,2}$ are the partial derivatives of ψ with respect to the first and second argument. It can be shown that (1.11) is exact if and only if

*See Definition 9.7.

$$M_{,2} = N_{,1} \quad \text{on } D. \tag{1.13}$$

Let us assume now that (1.13) is satisfied and choose a function $\psi : D \to \mathbb{R}$ such that (1.12) holds. If c is constant, I is an interval, and $x : I \to \mathbb{R}$ is a differentiable function such that $(t, x(t)) \in D$ and $\psi(t, x(t)) = c$ for all $t \in I$, then x is a solution of (1.11).

4. **Remark**: Sometimes an equation that is not of any of the forms discussed above can be converted to one of these forms by a simple device. Three such devices are mentioned below.

   (a) Sometimes a *substitution* or *change of variable* can be used to convert a nonlinear equation to a linear one or a nonseparable equation to a separable one.

   (b) Occasionally a nonlinear equation becomes linear if we interchange the roles of the independent and dependent variables. What this really amounts to is looking a differential equation for the *inverse function*.

   (c) In theory one can always find a nonzero function μ such that if we multiply equation (1.11) by μ it becomes exact. In practice, however, this approach is usually not of much use because it is very difficult to find a suitable μ.