Department of Mathematical Sciences
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21-476 **Ordinary Differential Equations** **Fall 2003**

III. Some Remarks on Uniform Convergence

Let $a, b \in \mathbb{R}$ with $a < b$ and a norm $\|\cdot\|$ on $\mathbb{R}^n$ be given. Let $\{y_{(m)}\}_{m=1}^{\infty}$ be a sequence of functions from $[a,b]$ to $\mathbb{R}^n$ and y be a function from $[a,b]$ to $\mathbb{R}^n$. Finally, let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers.

Recall that $y_{(m)} \to y$ *uniformly* on $[a,b]$ as $m \to \infty$ if there is a sequence $\{a_m\}_{m=1}^{\infty}$ of real numbers such that $a_m \to 0$ as $m \to \infty$ and

$$\|y_{(m)}(t) - y(t)\| \leq a_m \qquad \text{for all } t \in [a,b],\ m \in \mathbb{N}. \tag{3.1}$$

Lemma 3.1: *Assume that $y_{(m)}$ is continuous on $[a,b]$ for every $m \in \mathbb{N}$ and that $y_{(m)} \to y$ uniformly on $[a,b]$ as $m \to \infty$. Then y is continuous on $[a,b]$ and*

$$\int_a^b y_{(m)}(t)dt \to \int_a^b y(t)dt \ \text{ as } \ m \to \infty. \tag{3.2}$$

Lemma 3.2: *Assume that $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and that $y_{(m)}$ is continuous on $[a,b]$ for every $m \in \mathbb{N}$. Define $z_{(m)},\ z : [a,b] \to \mathbb{R}^n$ by*

$$z_{(m)}(t) = f(t, y_{(m)}(t)) \qquad \text{for all } t \in [a,b], m \in \mathbb{N}, \tag{3.3}$$

$$z(t) = f(t, y(t)) \qquad \text{for all } t \in [a,b]. \tag{3.4}$$

If $y_{(m)} \to y$ uniformly on $[a,b]$ as $m \to \infty$ then $z_{(m)} \to z$ uniformly on $[a,b]$ as $m \to \infty$ and consequently

$$\int_a^b f\left(t, y_{(m)}(t)\right) dt \to \int_a^b f(t, y(t))dt \ \text{ as } \ m \to \infty. \tag{3.5}$$

Ascoli-Arzela Theorem (Special Case): *Suppose that there exist $K, M \in \mathbb{R}$ such that*

$$\|y_{(m)}(t)\| \leq K \qquad \text{for all } t \in [a,b],\ m \in \mathbb{N}, \tag{3.6}$$

$$\|y_{(m)}(t) - y_{(m)}(s)\| \leq M|t-s| \qquad \text{for all } s,\ t \in [a,b],\ m \in \mathbb{N}. \tag{3.7}$$

Then $y_{(m)}$ *is continuous on* $[a,b]$ *for every* $m \in \mathbb{N}$ *and the sequence* $\left\{y_{(m)}\right\}_{m=1}^{\infty}$ *has a subsequence that converges uniformly on* $[a,b]$ *as* $m \to \infty$.

Theorem (Weierstrass M-Test): *Let* $\{M_m\}_{m=1}^{\infty}$ *be a sequence of real numbers such that*

$$\|y_{(m)}(t)\| \leq M_m \qquad \text{for all } t \in [a,b],\ m \in \mathbb{N}, \tag{3.8}$$

and define

$$S_{(m)}(t) = \sum_{k=1}^{m} y_{(k)}(t) \qquad \text{for all } t \in [a,b],\ m \in \mathbb{N}. \tag{3.9}$$

If $\displaystyle\sum_{m=1}^{\infty} M_m < \infty$ *then the sequence* $\left\{S_{(m)}\right\}_{m=1}^{\infty}$ *converges uniformly on* $[a,b]$.