Department of Mathematical Sciences
Carnegie Mellon University

21-476 **Ordinary Differential Equations** **Fall 2003**

Assignment 8

Due on Friday, December 5

1. Let $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $T > 0$ be given. Assume that A is continuous and $A(t+T) = A(t)$ for all $t \in \mathbb{R}$. Let X be the fundamental matrix solution of

   (LH) $$\dot{x}(t) = A(t)x(t)$$

   satisfying $X(0) = I$.

   (a) Show that (LH) has a nontrivial T-periodic solution if and only 1 is an eigenvalue of $X(T)$.

   (b) What is the situation regarding periodic solutions of (LH) if -1 is an eigenvalue of $X(T)$?

2. Assume that $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is continuous and satisfies $(A(t))^T = -A(t)$ for all $t \in \mathbb{R}$. Let X be the fundamental matrix solution of

   (LH) $$\dot{x}(t) = A(t)x(t)$$

   satisfying $X(0) = I$. Show that $(X(t))^{-1} = (X(t))^T$ for all $t \in \mathbb{R}$.

3. Assume that $q : \mathbb{R} \to \mathbb{R}$ is continuous and consider the second-order scalar equation

   (1) $$\ddot{u}(t) + q(t)u(t) = 0.$$

   If we let $x_1 = u,\ x_2 = \dot{u}$ then (1) can be rewritten as

   (2) $$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -q(t)x_1(t).\end{aligned}$$

   Show that if (2) has a fundamental matrix solution that is bounded on $[0,\infty)$ then no nontrivial solution of (2) can tend to zero as $t \to \infty$.

4. Assume that $p, q : \mathbb{R} \to \mathbb{R}$ are continuous and consider the second-order scalar equation

   (3) $$\ddot{u}(t) + p(t)\dot{u}(t) + q(t)u(t) = 0.$$

Let u_1, u_2 be solutions of (3) satisfying

$$u_1(0)\dot{u}_2(0) - \dot{u}_1(0)u_2(0) \neq 0$$

and let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ be given. Show that if $u_1(\alpha) = u_1(\beta) = 0$, then there is a $t^* \in (\alpha, \beta)$ such that $u_2(t^*) = 0$. (Suggestion: Suppose that no such t^* exists and consider the function $F : [\alpha, \beta] \to \mathbb{R}$ defined by $F(t) = \frac{u_1(t)}{u_2(t)}$ for all $t \in [\alpha, \beta]$.)