Supplementary Problems for Assignment 6

1. Let $n \in \mathbb{Z}^+$ be given. The citizens of a town of population n want to form clubs. There are two basic rules governing club membership:

   (a) Every club must have an odd number of members.

   (b) Every pair of distinct clubs must have an even number of members in common.

Show that there can be at most n clubs.

Suggestion: Let $\mathbb{F} = \mathbb{Z}_2$ and $V = \mathbb{F}^n$. Suppose that there are m clubs and consider the list $u_1, u_2, \ldots, u_m$ of vectors in V such that $(u_i)_j = 1$ if citizen j belongs to club i, and equals zero otherwise. The function $(\cdot, \cdot) : V \times V \to \mathbb{F}$ defined by

$$(x, y) = \sum_{i=1}^{n} x_i y_i$$

may be useful.

In Problems 2 through 6 let V be a real vector space with inner product $(\cdot, \cdot)$ and associated norm $||\cdot||$.

2. Show that

$$(x, y) = \frac{1}{4}(||x + y||^2 - ||x - y||^2) \quad \forall x, y \in V.$$

3. Let $u, v, \in V$ be given and assume that $||u|| = 6$, $||u+v|| = 4$, and $||u-v|| = 6$. Determine $||v||$ if possible.

4. Show that

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2) \quad \forall x, y \in V.$$

5. Let $\epsilon > 0$ be given. Show that

$$|(x, y)| \leq \epsilon ||x||^2 + \frac{1}{4\epsilon}||y||^2 \quad \forall x, y \in V.$$

6. $T \in L(V, V)$, $v_1, v_2 \in V$, and $\lambda_1, \lambda_2 \in \mathbb{R}$ with $||v_1|| = ||v_2|| = 1$ and $\lambda_1 \neq \lambda_2$ be given. Assume that $(Tx, y) = (x, Ty)$ for all $x, y \in V$ and that $Tv_1 = \lambda_1 v_1$, $Tv_2 = \lambda_2 v_2$. Show that $(v_1, v_2) = 0$.