Department of Mathematical Sciences
Carnegie Mellon University

21-476 **Ordinary Differential Equations** **Fall 2003**

Assignment 6

Due on Monday, November 17

1. Determine as much as you can about the stability of $(0,0,0)$ by studying Liapunov functions of the form $V(z_1,z_2,z_3) = az_1^2 + bz_2^2 + cz_3^2$.

   (a)
   $$\begin{aligned} \dot{x}_1 &= -2x_2 + x_2x_3 \\ \dot{x}_2 &= x_1 - x_1x_3 \\ \dot{x}_3 &= x_1x_2 \end{aligned}$$

   (b)
   $$\begin{aligned} \dot{x}_1 &= -2x_2 + x_2x_3 - x_1^3 \\ \dot{x}_2 &= x_1 - x_1x_3 - x_2^3 \\ \dot{x}_3 &= x_1x_2 - x_3^3 \end{aligned}$$

2. Assume that $\alpha, f : \mathbb{R} \to \mathbb{R}$ are continuously differentiable and define $F : \mathbb{R} \to \mathbb{R}$ by

   $$F(z) = \int_0^z f(\lambda)d\lambda \qquad \text{for all } z \in \mathbb{R}.$$

   Assume further that $\alpha(z) > 0$, for all $z \in \mathbb{R}$, $zf(z) > 0$ for all $z \in \mathbb{R}\backslash\{0\}$, and that $F(z) \to \infty$ as $|z| \to \infty$. Consider the second order equation

   $$\ddot{u} + \alpha(u)\dot{u} + f(u) = 0. \tag{1}$$

   If we let $x_1 = u,\ x_2 = \dot{u}$, then(1) can be rewritten as

   $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha(x_1)x_2 - f(x_1). \end{aligned}$$

   Show that for every $p \in \mathbb{R}^2$, $\varphi(t,p) \to 0$ as $t \to \infty$.

3. Let $f, c : \mathbb{R} \to \mathbb{R}$ and $\alpha : \mathbb{R}^2 \to \mathbb{R}$ be given and consider the system (of scalar equations)

   $$\begin{aligned} &\ddot{u} + f(u) + \alpha(u,\theta)\theta = 0 \\ &\dot{\theta} + c(\theta) - \alpha(u,\theta)\dot{u} = 0. \end{aligned} \tag{3}$$

If we let $x_1 = u,\ x_2 = \dot{u},\ x_3 = \theta$, then (3) can be rewritten as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -f(x_1) - \alpha(x_1, x_3)x_3 \\ \dot{x}_3 &= -c(x_3) + \alpha(x_1, x_3)x_2. \end{aligned} \tag{4}$$

Give conditions on f, c, α which ensure that for every $p \in \mathbb{R}^3,\ \varphi(t, p) \to 0$ as $t \to \infty$.

4. If you would like to turn in a solution (or a revision of a previous solution) to Problem 2 on Assignment 5 you may do so now.