Supplementary Problems for Assignment 4

Let $\mathbb{F}$ be a field and V, W be finite-dimensional vector spaces over $\mathbb{F}$.

1. Let $a_1, a_2, \ldots, a_n \in V$ and $T \in L(V, W)$ be given. Show that if $\text{span}(a_1, a_2, \ldots, a_n) = V$ and T is surjective then $\text{span}(Ta_1, Ta_2, \ldots, Ta_n) = W$.

2. Let $T \in L(V, V)$ be given. Prove that the following three statements are equivalent

   (a) T is invertible.

   (b) T is injective.

   (c) T is surjective.

   Do not use Theorem 13.9 from the book. You may use the result of Problem 8 from Test 1.

3. Let U be a subspace of V. Show that there exists $T \in L(V, V)$ such that $\{u \in V : Tu = 0\} = U$.

4. Show that there exists a surjective linear mapping $T \in L(V, W)$ if and only if $\dim W \leq \dim V$.

5. Show that there exists an injective linear mapping $T \in L(V, W)$ if and only if $\dim V \leq \dim W$.

6. Show $S, T \in L(V, V)$ be given. Show that the product ST is invertible if and only if both S and T are invertible.