Department of Mathematical Sciences
Carnegie Mellon University

21-476 **Ordinary Differential Equations** **Fall 2003**

Assignment 2

Due on Friday, September 19

1. Let $n = 1$ and consider the initial value problem

$$\dot{x}(t) = 2tx(t);\ x(0) = 1.$$

Find the Picard iterates $x_{(m)}$, $m \in \mathbb{N}$.

2. Let $n = 1$ and define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$(1) \qquad f(t,z) = \begin{cases} 0 & , \quad t \leq 0, \\ 2t & , \quad t > 0,\ z < 0, \\ 2t - \dfrac{4z}{t} & , \quad t > 0,\ 0 \leq z \leq t^2, \\ -2t & , \quad t > 0,\ z > t^2. \end{cases}$$

(Note that f is continuous, but is not locally Lipschitzean.)

Consider the initial value problem

$$(2) \qquad \dot{x}(t) = f(t, x(t));\ x(0) = 0.$$

Let $x_{(0)}(t) = 0$ for all $t \in \mathbb{R}$.

(a) Find the Picard iterates $x_{(m)}$, $m \in \mathbb{N}$.
(b) Show that for each $t > 0$, $\{x_{(m)}(t)\}_{m=1}^{\infty}$ diverges.
(c) Show that $\{x_{(2m-1)}\}_{m=1}^{\infty}$ and $\{x_{(2m)}\}_{m=1}^{\infty}$ converge uniformly on each bounded interval $I \subset \mathbb{R}$.
(d) Define $x^*,\ x_* : \mathbb{R} \to \mathbb{R}$ by

$$x^*(t) = \lim_{m\to\infty} x_{(2m-1)}(t),\ x_*(t) = \lim_{m\to\infty} x_{(2m)}(t).$$

for all $t \in \mathbb{R}$. Are $x^*,\ x_*$ solutions of (2)?

3. Let $n = 1$ and define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by (1). Let I be an interval containing 0 and suppose that $x^{(1)}$, $x^{(2)}$ are solutions of (2) on I. Show that $x^{(1)}(t) = x^{(2)}(t)$ for all $t \in I$.

4. Let $n = 1$ and consider the initial value problem

$$\dot{x}(t) = t^2 + x(t)^2; \ x(0) = 0.$$

   Put $x_{(0)}(t) = 0$ for all $t \in \mathbb{R}$.

   (a) Use Mathematica or Maple to compute the Picard iterates $x_{(1)},\ x_{(2)},\ x_{(5)},\ x_{(10)},\ x_{(20)}$.

   (b) Plot the graphs of $x_{(1)},\ x_{(2)},\ x_{(5)},\ x_{(10)},\ x_{(20)}$ on $[0, 1]$.

   (c) Plot the graphs of $x_{(1)},\ x_{(2)},\ x_{(5)},\ x_{(10)},\ x_{(20)}$ on $[0, 2]$.

5. Let $n = 2$ and consider the autonomous system

$$\begin{cases} \dot{x}_1 = \sin x_1 + x_2^3 \\ \dot{x}_2 = -x_1^3 x_2^2 - e^{-x_1} x_2. \end{cases} \tag{3}$$

   Let $\delta > 0$ be given and assume that x is a solution of (3) on $[0, \delta)$. Show that x is bounded on $[0, \delta)$. [Suggestion: Compute $\dot{\varphi}$ for the function $\varphi : [0, \delta) \to \mathbb{R}$ defined by

$$\varphi(t) = \frac{1}{4} x_1(t)^4 + \frac{1}{2} x_2(t)^2 \qquad \text{for all } t \in [0, \delta).]$$