Department of Mathematical Sciences
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21-476 **Ordinary Differential Equations** **Fall 2003**

VI. Autonomous Systems: Orbits and Limit Sets

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be given and consider the autonomous system

$$\text{(A)} \qquad \dot{x} = g(x).$$

We assume throughout this section that g is continuous and has the uniqueness property. For each $p \in \mathbb{R}^n$, let $(\eta_-(p), \eta_+(p))$ be the domain of the unique noncontinuable solution of

$$\text{(6.1)} \qquad \dot{x} = g(x);\ x(0) = p,$$

and for each $t \in (\eta_-(p),\ \eta_+(p))$ we write $\varphi(t, p)$ for the value at time t of the solution of (6.1). The fact that the time does not appear explicitly on the right-hand side of (A) has the following important consequence: *Let x be a solution of (A) and let $\tau \in \mathbb{R}$ be given. Then, the function y defined by $y(t) = x(t + \tau)$ is also a solution of (A).*

By combining this observation with the uniqueness property we obtain the following fundamental result.

Proposition 6.1: *Let $p \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $\tau \in (\eta_-(p),\ \eta_+(p))$ be given. Then $t + \tau \in (\eta_-(p), \eta_+(p))$ if and only if $t \in (\eta_-(\varphi(\tau, p)), \eta_+(\varphi(\tau, p)))$. Moreover, if $t + \tau \in (\eta_-(p),\ \eta_+(p))$ then*

$$\text{(6.2)} \qquad \varphi(t + \tau, p) = \varphi(t, \varphi(\tau, p)).$$

Definition 6.2: *A point $p \in \mathbb{R}^n$ is called a critical point or equilibrium point for (A) if $g(p) = 0$.*

Definition 6.3: *Let $p \in \mathbb{R}^n$ be given. The orbit through p is the curve $\gamma(p)$ defined by $\gamma(p) = \{\varphi(t, p) : \eta_-(p) < t < \eta_+(p)\}$.*

It is useful to decompose each orbit into two pieces – one corresponding to positive times and the other to negative times.

Definition 6.4: *Let $p \in \mathbb{R}^n$ be given. The positive semiorbit through p and the negative semiorbit through p are defined by*

$$\gamma^+(p) = \{\varphi(t,p) : 0 \le t < \eta_+(p)\},$$
$$\gamma^-(p) = \{\varphi(t,p) : \eta_-(p) < t \le 0\},$$

respectively.

Notice that for every $p \in \mathbb{R}^n$ we have $p \in \gamma^+(p)$, $p \in \gamma^-(p)$ and $\gamma(p) = \gamma^-(p) \cup \gamma^+(p)$.

Definition 6.5: *Let $S \subset \mathbb{R}^n$. We say that S is*

(i) *invariant if $\gamma(p) \subset S$ for every $p \in S$;*

(ii) *positively invariant if $\gamma^+(p) \subset S$ for every $p \in S$;*

(iii) *negatively invariant if $\gamma^-(p) \subset S$ for every $p \in S$.*

Notice that every orbit is invariant, every positive semiorbit is positively invariant, and every negative semiorbit is negatively invariant.

Proposition 6.6: *Let $p, q \in \mathbb{R}^n$ be given. If $\gamma(p) \cap \gamma(q) \neq \emptyset$ then $\gamma(p) = \gamma(q)$.*

Proposition 6.7: *Let $p \in \mathbb{R}^n$ be given and assume that $g(p) \neq 0$. Then $\gamma(p)$ is a closed curve if and only if $\eta_-(p) = -\infty, \eta_+(p) = \infty$, and there exists $T > 0$ such that $\varphi(t+T,p) = \varphi(t,p)$ for all $t \in \mathbb{R}$.*

Definition 6.8: *Let $p \in \mathbb{R}^n$ be given. We say that $\gamma(p)$ is a periodic orbit if $g(p) \neq 0$ and $\gamma(p)$ is a closed curve.*

Remark 6.9: *Let $p \in \mathbb{R}^n$ be given and assume that $\gamma(p)$ is a periodic orbit. Let T be the smallest strictly positive time such that $\varphi(T,p) = p$. Then $\gamma(p) = \{\varphi(t,p) : 0 \le t \le T\}$ and $\varphi(t,p) \neq \varphi(s,p)$ for all $s,t \in [0,T)$ with $s \neq t$. In particular, $\gamma(p)$ is a simple closed curve.*

Definition 6.10: *Let $p, q \in \mathbb{R}^n$ be given. We say that q is an ω-limit point for p if $\eta_+(p) = \infty$ and there is a sequence $\{t_m\}_{m=1}^{\infty}$ of times such that $t_m \ge 0$ for all $m \in \mathbb{N}$, $t_m \to \infty$ as $m \to \infty$ and $\varphi(t_m,p) \to q$ as $m \to \infty$. We say that q is an α-limit point for p if $\eta_-(p) = -\infty$ and there is a sequence of times $\{t_m\}_{m=1}^{\infty}$ such that $t_m \le 0$ for all $m \in \mathbb{N}$, $t_m \to -\infty$ as $m \to \infty$ and $\varphi(t_m,p) \to q$ as $m \to \infty$. The set of all ω-limit points for p is called the ω-limit set for p and is denoted $\omega(p)$. The set of all α-limit points for p is called the α-limit set for p and is denoted $\alpha(p)$.*

Theorem 6.11: *Let $p \in \mathbb{R}^n$ be given and assume that $\gamma^+(p)$ is bounded. Then $\omega(p)$ is nonempty, closed, bounded, and invariant. Moreover, $dist(\varphi(t,p), \omega(p)) \to 0$ as $t \to \infty$. Finally, $\omega(p)$ cannot be expressed as a union of two nonempty closed disjoint sets.*

Definition 6.12: *Let $V : \mathbb{R}^n \to \mathbb{R}$ be given and assume that V is continuously differentiable. We define $\dot{V} : \mathbb{R}^n \to \mathbb{R}$ by*

$$\dot{V}(z) = \nabla V(z) \cdot g(z) \qquad \text{for all } z \in \mathbb{R}^n.$$

Remark 6.13: *If* $V : \mathbb{R}^n \to \mathbb{R}$ *is continuously differentiable and* x *is a solution of (A) then*

$$\frac{d}{dt}V(x(t)) = \dot{V}(x(t)) \qquad \text{for all } t \in \text{Dom}(x)$$

by virtue of the chain rule.

Theorem 6.14 (Invariance Principle): *Let* $p \in \mathbb{R}^n$ *and* $V : \mathbb{R}^n \to \mathbb{R}$ *be given. Assume that* V *is continuously differentiable and that* $\dot{V}(z) \leq 0$ *for all* $z \in \mathbb{R}^n$*. Let* $S = \{z \in \mathbb{R}^n : \dot{V}(z) = 0\}$*. If* $\gamma^+(p)$ *is bounded then* $\omega(p) \subset S$*.*

Proposition 6.15: *Assume that* $V : \mathbb{R}^n \to \mathbb{R}$ *is continuously differentiable and satisfies* $V(z) \to \infty$ *as* $||z|| \to \infty$*. If* $\dot{V}(z) \leq 0$ *for all* $z \in \mathbb{R}^n$*, then* $\gamma^+(p)$ *is bounded for each* $p \in \mathbb{R}^n$*.*

Definition 6.16: Let S and T be subsets of $\mathbb{R}^n$. We say that S and T are homeomorphic if there is a bijective mapping $F : S \to T$ such that F and F^{-1} are continuous.

Theorem 6.17: Let S be a subset of $\mathbb{R}^n$ that is positively invariant and homeomorphic to $\{z \in \mathbb{R}^n : ||z|| \leq 1\}$. Then S contains a critical point.