Department of Mathematical Sciences
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21-476 **Ordinary Differential Equations** **Fall 2003**

VII. Positive Definite Functions

Recall that a real $n \times n$ matrix A is said to be *positive definite* if

$$z \cdot (Az) = \sum_{i,j=1}^{n} A_{ij} z_i z_j > 0 \qquad \text{for all } z \in \mathbb{R}^n \backslash \{0\}.$$

Recall also that if a real $n \times n$ matrix A is symmetric (i.e. if $A = A^T$) then all eigenvalues of A are real and there is an orthormal basis for $\mathbb{R}^n$ consisting solely of eigenvectors of A.

Proposition 7.1: Let A be a real, symmetric $n \times n$ matrix. For each $k = 1, 2 \ldots n$, define the $k \times k$ matrix $A^{(k)}$ by $A^{(k)}_{ij} = A_{ij}$ for $i, j = 1, 2, \ldots k$. The following three statements are equivalent.

(i) A is positive definite.

(ii) All eigenvalues of A are strictly positive.

(iii) $\det \left(A^{(k)}\right) > 0$ for each $k = 1, 2, \ldots, n$.

Definition 7.2:

Let $x^* \in \mathbb{R}^n$ and $V : (\text{Dom}(V) \subset \mathbb{R}^n) \to \mathbb{R}$ be given. We say that V is

(a) locally positive definite at x^* if there is an open set U such that $x^* \in U \subset \text{Dom}(V), V(x^*) = 0$, and $V(z) > 0$ for all $z \in U \backslash \{x^*\}$.

(b) locally negative definite at x^* if $-V$ is positive definite at x^*.

(c) globally positive definite at x^* if $\text{Dom}(V) = \mathbb{R}^n$, $V(x^*) = 0$, and $V(z) > 0$ for all $z \in \mathbb{R}^n \backslash \{x^*\}$.

(d) globally negative definite at x^* if $-V$ is globally positive definite at x^*.

(e) locally negative semidefinite at x^* if there is an open set U such that $x^* \in U \subset \text{Dom}(V)$, $V(x^*) = 0$, and $V(z) \leq 0$ for all $z \in U$.

(f) globally negative semidefinite at x^* if $\text{Dom}(V) = \mathbb{R}^n$, $V(x^*) = 0$ and $V(z) \leq 0$ for all $z \in \mathbb{R}^n$.

Theorem 7.3: Let $x^* \in \mathbb{R}^n$, an open set U with $x^* \in U \subset \mathbb{R}^n$, and $V : U \to \mathbb{R}$ be given. Assume that V has continuous second order partial derivatives and define the $n \times n$ matrix $H(x^*)$ by

$$(H(x^*))_{ij} = V_{,i,j}(x^*)$$

[Note the $H(x^*)$ is symmetric by equality of mixed partials.] Assume that $V(x^*) = 0$, $\nabla V(x^*) = 0$, and that $H(x^*)$ is positive definite. Then V is locally positive definite at x^*.

Corollary 7.4: Let $x^* \in \mathbb{R}^2$, an open set U with $x^* \in U \subset \mathbb{R}^2$, and $V : U \to \mathbb{R}$ be given. Assume that V has continuous second-order partial derivatives and put $\alpha = V_{,1,1}(x^*)$, $\beta = V_{,1,2}(x^*)$, and $\gamma = V_{,2,2}(x^*)$. Assume that $V(x^*) = V_{,1}(x^*) = V_{,2}(x^*) = 0$.

(i) If $\alpha > 0$ and $\alpha\gamma - \beta^2 > 0$ then V is locally positive definite at x^*.

(ii) If $\alpha < 0$ and $\alpha\gamma - \beta^2 > 0$ then V is locally negative definite at x^*.