21-355 **Principles of Real Analysis I** **Fall 2004**

Solutions to Assignment 7

3. Assume that g is twice differentiable, $g(r) = 0$, $g'(x) > 0$ and $g''(x) > 0$ for all $x \in \mathbb{R}$. Notice that $g(x) > 0$ if and only if $x > r$. We shall make use of the following proposition proved in lecture.

   Prop: $z - \frac{g(z)}{g'(z)} > r \quad \forall z \in (r, \infty)$.

   Claim: $r < x_{n+1} < x_n \quad \forall n \in \mathbb{N}$.

   The claim will be proved by induction

   Base Case: $x_1 > x_2 > r$ by assumption.

   Inductive Step: Let $k \in \mathbb{N}$ be given and assume that

$$r < x_{k+1} < x_k.$$

   By the mean value theorem we may choose $c_k \in (x_{k+1}, x_k)$ such that

$$g'(c_k) = \frac{g(x_{k+1}) - g(x_k)}{x_{k+1} - x_k}.$$

   Since $g'' > 0$ and $c_k > x_{k+1}$, it follows that $g'(c_k) > g'(x_{k+1})$. Notice that

$$x_{k+2} = x_{k+1} - \frac{g(x_{k+1})}{g'(c_k)}.$$

   Since $g'(x_{k+1}) > 0$ and $g(x_{k+1}) > 0$ we deduce that

$$x_{k+2} > x_{k+1} - \frac{g(x_{k+1})}{g'(x_{k+1})}.$$

   The proposition yields $x_{k+2} > r$. Since $g(x_{k+1}) > 0$ and $g'(c_k) > 0$ we conclude that

$$x_{k+2} - x_{k+1} = -\frac{g(x_{k+1})}{g'(c_k)} < 0$$

   so that $r < x_{k+2} < x_{k+1}$. $\square$

   The sequence $\{x_n\}_{n=1}^{\infty}$ is decreasing and bounded below, and therefore convergent. Let $L = \lim_{n\to\infty} x_n$. For each $n \in \mathbb{N}$, we choose $c_n \in (x_{n+1}, x_n)$ such that

$$g'(c_n) = \frac{g(x_{n+1}) - g(x_n)}{x_{n+1} - x_n},$$

and notice that $c_n \to L$ as $n \to \infty$ by the squeeze theorem. Using the recursion relation and the continuity of g and g' we find that

$$L = L - \frac{g(L)}{g'(L)},$$

which yields $g(L) = 0$. It follows that $L = r$ and $x_n \to r$ as $n \to \infty$.

4. Notice that

$$f_n(x) - \alpha_n < g(x) < f_n(x) + \alpha_n \quad \forall n \in \mathbb{N},\ x \in [a, b].$$

Let $P \in \mathcal{P}[a, b]$ be given. Then we have

$$\begin{array}{ll} m_i(g) \geq m_i(f_n) - \alpha_n & \text{and} \\ M_i(g) \leq M_i(f_n) + \alpha_n & \end{array}$$

so that

$$U(g, P) - L(g, P) \leq 2\alpha_n + U(f_n, P) - L(f_n, P).$$

Let $\epsilon > 0$ be given and choose $N \in \mathbb{N}$ such that $|\alpha_N| < \epsilon/4$. Since $f_N \in \mathcal{R}[a, b]$ we may choose $P_N \in \mathcal{P}[a, b]$ such that

$$U(f_N, P_n) - L(f_N, P_N) < \epsilon/2.$$

It follows that

$$U(g, P) - L(g, P) \leq 2\alpha_N + U(f_N, P_N) - L(f_N, P_N) < \epsilon/2 + \epsilon/2$$

and $g \in \mathcal{R}[a, b]$.

To prove the final claim, observe that

$$\begin{array}{ll} \left| \int_a^b g - \int_a^b f_n \right| & = \left| \int_a^b (g - f_n) \right| \\ & \leq \int_a^b |g - f_n| \\ & \leq \int_a^b \alpha_n = \alpha_n(b - a). \end{array}$$

Since $\alpha_n(b - a) \to 0$ as $n \to \infty$ we conclude that

$$\int_a^b g = \lim_{n \to \infty} \int_a^b f_n.$$

5. Since $\mathcal{R}[a,b] \subset \mathcal{B}[a,b]$, we may choose $M > 0$ such that

$$|f(t)| \leq M \quad \forall t \in [a,b].$$

Let $\epsilon > 0$ be given and put $\delta = \epsilon/M$. Then for all $x, y \in [a,b]$ with $|x-y| < \delta$ we have

$$\begin{aligned} |F(x) - F(y)| &= \left| \int_a^x f(t)dt - \int_a^y f(t)dt \right| \\ &= \left| \int_y^x f(t)dt \right| \\ &\leq \left| \int_y^x M dt \right| \leq M\,|x-y| \\ &< M\left(\tfrac{\epsilon}{M}\right) = \epsilon \end{aligned}$$

6. Let $A = \int_a^b g^2$, $B = \int_a^b fg$, and $C = \int_a^b f^2$. then

$$\begin{aligned} H(\lambda) = \ & \int_a^b (f(x) - \lambda g(x))^2 dx \\ & = C - 2B\lambda + A\lambda^2 \\ & \geq 0 \quad \text{for all } \lambda \in \mathbb{R} \end{aligned}$$

Case 1: $A \neq 0$.

Then the quadratic equation $A\lambda^2 - 2B\lambda + C = 0$ has at most one real root, so that $4B^2 - 4AC \leq 0$. This yields $B \leq A^{1/2}C^{1/2}$ which is the desired inequality.

Case 2: $A = 0$.

Then the linear expression $C - 2B\lambda$ is always nonnegative. This implies $B = 0$ which is the desired inequality when $A = 0$.