Solutions to Assignment 5

1. **Claim 1**: f is discontinuous at each $z \in \mathbb{Q} \cap [0,1]$.

   Proof of Claim 1: Let $z \in \mathbb{Q} \cap [0,1]$ be given and note that $f(z) > 0$. Since $cl([0,1] \backslash \mathbb{Q}) = [0,1]$, we may choose a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in [0,1] \backslash \mathbb{Q}$ for every $n \in \mathbb{N}$ and $x_n \to z$ as $n \to \infty$. Since $f(x_n) = 0$ for every $n \in \mathbb{N}$, we have $\lim_{n\to\infty} f(x_n) = 0 \neq f(z)$. It follows that f is discontinuous at z.

   Claim 2: f is continuous at each $y \in [0,1] \backslash \mathbb{Q}$.

   The proof of Claim 2 will make use of the following lemma.

   Lemma: Let $y \in [0,1] \backslash \mathbb{Q}$ and $N \in \mathbb{N}$ be given. There exists $\delta > 0$ such that $q(x) > N$ for all $x \in B_\delta(y) \cap \mathbb{Q} \cap [0,1]$.

   Proof of Lemma: For each $n \in \mathbb{N}$ let $D_n = \{x \in \mathbb{Q} \cap [0,1] : q(x) = n\}$. Notice that $D_1 = \{0, 1\}$, and for each $n \geq 2$, D_n contains at most $n-1$ elements. Let $A_n = \{x \in \mathbb{Q} \cap [0,1] : q(x) \leq N\}$ and notice that

   $$A_N = \bigcup_{n=1}^{N} D_n.$$

   It follows that A_N is nonempty and finite. Put

   $$\delta = \min \{|y - x| : \; x \in A_N\}$$

   and notice that $\delta > 0$. Now, let $x \in B_\delta(y) \cap \mathbb{Q} \cap [0,1]$ be given. Since $|x - y| < \delta$, it follows that $x \notin A_N$ and consequently $q(x) > N$. □

   Proof of Claim 2: Let $y \in [0,1] \backslash \mathbb{Q}$ and $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ with $N > \frac{1}{\epsilon}$. Now choose $\delta > 0$ as in the lemma. Let $x \in B_\delta(y) \cap [0,1]$ be given. If $x \in [0,1] \backslash \mathbb{Q}$ then

   $$|f(x) - f(y)| = |0 - 0| = 0 < \epsilon.$$

   If $x \in \mathbb{Q}$ then $q(x) > N$ so that

   $$|f(x) - f(y)| = \frac{1}{q(x)} < \frac{1}{N} < \epsilon.$$

   It follows that f is continuous at y.

4 (a) Assume that f and g are bounded on S. Choose $M_1, M_2 > 0$ such that

$$|f(x)| \leq M_1, \ |g(x)| \leq M_2 \quad \forall x \in S.$$

Let $\epsilon > 0$ be given. Since f, g are uniformly continuous on S we may choose $\delta_1, \delta_2 > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{2M_2} \quad \forall x, y \in S, \ |x - y| < \delta_1$$

$$|g(x) - g(y)| < \frac{\epsilon}{2M_1} \quad \forall x, y \in S |x - y| < \delta_2,$$

and put $\delta = \min\{\delta_1, \delta_2\}$.
Then, for all $x, y \in S$ with $|x - y| < \delta$ we have

$$\begin{aligned} |F(x) \ &-F(y)| = |f(x)g(x) - f(y)g(y)| \\ &\leq |f(x)(g(x) - g(y))| + |g(y)(f(x) - f(y))| \\ &\leq M_1 |g(x) - g(y)| + M_2 |f(x) - f(y)| \\ &< M_1 \left(\frac{\epsilon}{2M_1} \right) + M_2 \left(\frac{\epsilon}{2M_2} \right) = \epsilon. \end{aligned}$$

It follows that F is uniformly continuous on S.

(b) If f is bounded, but g is not, then F need not be uniformly continuous. As an example, take $S = \mathbb{R}$,

$$f(x) = \frac{\sin(x^2)}{1 + |x|} \quad \forall x \in \mathbb{R}$$

$$g(x) = 1 + |x| \quad \forall x \in \mathbb{R}.$$

Then g is uniformly continuous (take $\delta = \epsilon$), and f is uniformly continuous by Problem 4 on Assignment 4. Notice that

$$F(x) = f(x)g(x) = \sin(x^2) \quad \forall x \in \mathbb{R}.$$

We showed that F is not uniformly continuous in one of the problem sessions.

6. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x) - \alpha x$ for all $x \in \mathbb{R}$. Notice that g is differentiable on $\mathbb{R}$ and $g'(x) = f'(x) - \alpha$ for all $x \in \mathbb{R}$. It follows that $g'(a) < 0$ and $g'(b) > 0$. Since g is continuous on $[a, b]$, we may choose $c \in [a, b]$ such that $g(c) \leq g(x)$ for all $x \in [a, b]$. Notice that $c \neq a$, because if we had $g(x) \geq g(a)$ for all $x \in [a, b]$, it would follow that $g'(a) \geq 0$. A similar argument gives $c \neq b$. Consequently, $c \in (a, b)$ and $g'(c) = 0$. It follows that $f'(c) = \alpha$.

7. Assume that f is differentiable on $\mathbb{R}$ and that f' is bounded. Choose $M > 0$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. For $x, y \in \mathbb{R}$ with $x \neq y$ we may choose $C_{x,y}$ between x and y such that

$$f(x) - f(y) = f'(C_{x,y})(x - y)$$

by virtue of the mean value theorem. It follows that

$$\begin{aligned} |f(x) - f(y)| &= |f'(C_{x,y})| \cdot |x - y| \\ &\leq M|x - y| \\ &\forall x, y \in \mathbb{R},\ x \neq y. \end{aligned}$$

If $x = y$, then $f(x) - f(y) = x - y = 0$, and consequently

$$(*) \qquad |f(x) - f(y)| \leq M|x - y| \quad \forall x, y \in \mathbb{R}.$$

Let $\epsilon > 0$ be given and put $\delta = \frac{\epsilon}{M}$. Then, for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$ we have

$$|f(x) - f(y)| \leq M|x - y| < M\left(\frac{\epsilon}{M}\right) = \epsilon.$$