Solutions to Assignment 4

2. Since f is uniformly continuous we may choose $\delta > 0$ such that

$$|f(x) - f(y)| < 1 \qquad \forall x, y \in (0,1) \text{ with } |x - y| < \delta.$$

Put $\delta^* = \min\{\frac{1}{3}, \delta\}$. Since f is continuous and $[\delta^*, 1-\delta^*]$ is nonempty and compact, f attains a maximum and a minimum on $[\delta^*, 1-\delta^*]$. Therefore, we may choose $M^* \in \mathbb{R}$ such that

$$|f(x)| \leq M^* \quad \forall x \in [\delta^*, 1-\delta^*].$$

Claim: $|f(x)| \leq M^* + 1 \quad \forall x \in (0,1)$.

To prove the claim, let $x \in (0,1)$ be given and notice that exactly one of the following must hold:

Case 1: $x \in [\delta^*, 1-\delta^*]$;

Case 2: $x \in (0, \delta^*)$;

Case 3: $x \in (1-\delta^*, 1)$.

If $x \in [\delta^*, 1-\delta^*]$ then $|f(x)| \leq M^* \leq M^* + 1$.

If $x \in (0, \delta^*)$ then $|x - \delta^*| < \delta^* \leq \delta$ so that

$$|f(x)| \leq |f(1-\delta^*, 1)| + |f(x) - f(\delta^*)| \leq M^* + 1$$

If $x \in (1-\delta^*, 1)$ then $|x - (1-\delta^*)| < \delta^* \leq \delta$ so that

$$|f(x)| \leq |f(1-\delta^*)| + |f(x) - f(1-\delta^*)| \leq M^* + 1. \qquad \square$$

3. Since $f(y) > 0$ and f is continuous at y, we may choose $\delta > 0$ such that

$$|f(x) - f(y)| < f(y) \quad \forall x \in B_\delta(y) \cap S.$$

It follows that

$$-f(y) < f(x) - f(y) < f(y) \quad \forall x \in B_\delta(y) \cap S.$$

which yields

$$0 < f(x) < 2f(y) \quad \forall x \in B_\delta(y)n \cap S.$$

4. Let $\epsilon > 0$ be given. Choose $M > 0$ such that

$$|f(x)| < \epsilon/2 \quad \forall x \in \mathbb{R},\ |x| > M.$$

Notice that $[-M-1,\ M+1]$ is compact. Since f is continuous, the restriction of f to $[-M-1,\ M+1]$ is uniformly continuous. Therefore, we may choose $\delta_1 > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \forall x, y \in [-M-1,\ M+1],\ |x - y| > \delta_1.$$

Put $\delta = \min\{1, \delta_1\}$.

Claim: $|f(x) - f(y)| < \epsilon \quad \forall x, y \in \mathbb{R},\ |x - y| < \delta$.

To prove the claim, let $x, y \in \mathbb{R}$ with $|x - y| < \delta$ be given. Notice that one of the following must hold:

Case 1: $x, y \in [-M-1,\ M+1]$;

Case 2: $x, y > M$;

Case 3: $x, y < -M$.

If $x, y \in [-M-1,\ M+1]$ then $|f(x) - f(y)| < \epsilon$ since $\delta \leq \delta^*$.
If $x, y > M$ then

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| < \epsilon/2 + \epsilon/2.$$

Similarly, if $x, y < -M$ then

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| < \epsilon/2 + \epsilon/2. \qquad \square$$

7. Let $y \in S$ be given. Then $y \in cl(T)$ so we may choose a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S \quad \forall n \in \mathbb{N}$ and $x_n \to y$ as $n \to \infty$. Notice that $f(x_n) = g(x_n) \quad \forall n \in \mathbb{N}$. Since f and g are continuous at y, it follow that $f(x_n) \to f(y)$ and $g(x_n) \to g(y)$ as $n \to \infty$. We conclude that $f(y) = g(y)$. $\square$