21-355 **Principles of Real Analysis I** **Fall 2004**

Solutions to Assignment 3

2. Let $\mathcal{C}$ be a collection of open sets that covers S. Choose $\mathcal{O}_0 \in S$ such that $O \in \mathcal{O}_0$ and then choose $\delta > 0$ such that $B_\delta(0) \subset \mathcal{O}_0$. Now let $T = \{\frac{1}{n} : n \in \mathbb{N},\ n \leq \frac{1}{\delta}\}$. Observe that $T \subset S,\ T$ is finite and

$$S \backslash T \subset B_\delta(0) \subset \mathcal{O}_0.$$

For each $x \in T$, we choose $\mathcal{O}_x \in \mathcal{C}$ with $x \in \mathcal{O}_x$. Let

$$\mathcal{F} = \{\mathcal{O}_0\} \cup \{\mathcal{O}_x : x \in T\}.$$

Then $\mathcal{F}$ is a finite subcollection of $\mathcal{C}$ and $\mathcal{F}$ covers S since $T \subset \underset{x \in T}{\cup} \mathcal{O}_x$ and $S \backslash T \subset \mathcal{O}_0$. $\square$

3. Assume that S is nonempty and bounded above. Let $c = \sup(S)$. Let $\delta > 0$ be given. Then c is an upper bound for S and $c - \delta$ is not an upper bound for S. Therefore we may choose $x \in S$ with $c - \delta < x \leq c$. It follows that $x \in B_\delta(c) \cap S$ and $B_\delta(c) \cap S \neq \phi$. Since $\delta > 0$ was arbitrary, we conclude that $c \in cl(S)$. Since S is closed, it follows that $c \in S$. $\square$

4. We shall show that $\text{int}\ (S^c) \subset (cl(S))^c$ and $(cl(S))^c \subset \text{int}\ (S^c)$. Let $x_0 \in \text{int}\ (S^c)$ be given. Choose $\delta > 0$ such that $B_\delta(x_0) \subset S^c$. It follows that $B_\delta(x_0) \cap S = \phi$ which implies $x_0 \notin cl(S)$, i.e. $x_0 \in (cl(S))^c$. Let $x_0 \in (cl(S))^c$ be given. Since $x_0 \notin cl(S)$ we may choose $\delta > 0$ such that $B_\delta(x_0) \cap S = \phi$. It follows that $B_\delta(x_0) \subset S^c$ and $x_0 \in \text{int}\ (S^c)$. $\square$

   (a) Choose $k \in \mathbb{N}$ such that S_k is bounded. For each $n \in \mathbb{N}$, we choose $x_n \in S_n$. Since $x_n \in S_k$ for all $n \geq k$ and S_k is bounded, it follows that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. By the Bolzano-Weierstrass Theorem we may choose a convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$. Let $l = \lim\limits_{j \to \infty} x_{n_j}$. We shall show that $l \in \bigcap\limits_{n=1}^{\infty} S_n$. Let $m \in \mathbb{N}$ be given. Since S_m is closed and $x_n \in S_m$ for all $n \in \mathbb{N}$ with $n \geq m$, it follows from a minor variant of Proposition III.6 that $l \in S_m$. Since $m \in \mathbb{N}$ was arbitrary, we conclude that $l \in \bigcap\limits_{n=1}^{\infty} S_n$.

   (b) Put $S_n = [n, \infty)$ for every $n \in \mathbb{N}$.