Solutions to Assignment 2

3.a) Claim 1: $x_n > 0$ for all $n \in \mathbb{N}$.
Proof of Claim 1 (by induction):
Base Case: $x_1 = 2 > 0$

Inductive Step: Let $k \in \mathbb{N}$ be given and assume that $x_k > 0$. Then, $\frac{x_k}{2} > 0$ and $\frac{1}{x_k} > 0$ so that

$$x_{k+1} = \frac{x_k}{2} + \frac{1}{x_k} > 0.$$

Claim 2: $x_n^2 \geq 2$ for all $n \in \mathbb{N}$.
Proof of Claim 2: Let $n \in \mathbb{N}$ with $n \geq 2$ be given. Then we have

$$\begin{aligned} x_n^2 - 2 &= \left(\frac{x_{n-1}}{2} + \frac{1}{x_{n-1}}\right)^2 - 2 \\ &= \frac{x_{n-1}^2}{4} + 1 + \frac{1}{x_{n-1}^2} - 2 \\ &= \frac{x_{n-1}^2}{4} - 1 + \frac{1}{x_{n-1}^2} \\ &= \left(\frac{x_{n-1}}{2} - \frac{1}{x_{n-1}}\right)^2 \geq 0, \end{aligned}$$

so that $x_n^2 \geq 2$. Since $x_1^2 = 4 > 2$, the claim follows.

Claim 3: $x_{n+1} - x_n \leq 0$ for all $n \in \mathbb{N}$.

Proof of Claim 3: Let $n \in \mathbb{N}$ be given. Then

$$\begin{aligned} x_{n+1} - x_n &= \frac{x_n}{2} + \frac{1}{x_n} - x_n \\ &= \frac{1}{x_n} - \frac{x_n}{2} = \frac{2 - x_n^2}{2x_n} \\ &\leq 0 \end{aligned}$$

by virtue of Claims 1 and 2. Since $\{x_n\}_{n=1}^{\infty}$ is decreasing and bounded below it is convergent.

b) Let $l = \lim_{n\to\infty} x_n$. Notice that $l \geq \sqrt{2}$ (since $x_n \geq \sqrt{2}$ for all $n \in \mathbb{N}$). Notice also that $x_{n+1} \to l$, $\frac{x_n}{2} \to \frac{l}{2}$, and $\frac{1}{x_n} \to \frac{1}{l}$ as $n \to \infty$. Therefore,

$$l = \frac{l}{2} + \frac{1}{l},$$

which yields $l^2 = \frac{l^2}{2} + 1$, so that $l^2 = 2$. Since $l \geq \sqrt{2}$, we deduce that $l = \sqrt{2}$.

5. Let $\epsilon > 0$ be given. Choose N_1 and $N_2 \in \mathbb{N}$ such that

$$|x_n - l| < \epsilon \quad \forall n \in \mathbb{N}, \quad n \geq N_1,$$

$$|z_n - l| < \epsilon \quad \forall n \in \mathbb{N}, \quad n \geq N_2,$$

and put $N = \max\{N_1, N_2\}$. Then, for all $n \in \mathbb{N}$ with $n \geq N$ we have

$$-\epsilon < x_n - l < \epsilon \quad \textit{and}$$

$$-\epsilon < z_n - l < \epsilon.$$

Notice that

$$x_n - l \leq y_n - l \leq z_n - l \quad \forall n \in \mathbb{N}$$

by virtue of our assumptions. Therefore, we have

$$-\epsilon < x_n - l \leq y_n - l \leq z_n - l < \epsilon \quad \forall n \in \mathbb{N},\ n \geq N$$

which yields

$$|y_n - l| < \epsilon \quad \forall n \in \mathbb{N}, \quad n \geq N.$$

6. Let $\epsilon > 0$ be given and put $S_\epsilon = \{n \in \mathbb{N} : \ |x_n| < \epsilon\}$. We want to show that S_ϵ is infinite. We assume first that $\epsilon \leq 1$. Let $T_\epsilon = \{n \in \mathbb{N} : \ |x_n - \frac{\epsilon}{2}| < \epsilon/2\}$ and notice that T_ϵ is infinite since $\epsilon/2$ is cluster point. Observe that $T_\epsilon \subset S_\epsilon$. Indeed for a given $k \in \mathbb{N}$: if $k \in T_\epsilon$ then $|x_k| \leq \epsilon/2 + |x_k - \epsilon/2| < \epsilon$, so $k \in S_\epsilon$. Since S_ϵ has an infinite subset, we conclude that S_ϵ is infinite. Finally, if $\epsilon > 1$ then $\{n \in \mathbb{N} : \ |x_n| < 1\} \subset S_\epsilon$. Therefore S_ϵ has an infinite subset by the argument above.

7. Since $\mathbb{Q}$ is countably infinite, we may choose a bijection $g : \mathbb{N} \to \mathbb{Q}$. Define the sequence $\{x_n\}_{n=1}^\infty$ by $x_n = g(n)$ for all $n \in \mathbb{N}$.

   Claim: Let $l \in \mathbb{R}$ be given. Then l is a cluster point of $\{x_n\}_{n=1}^\infty$.

Proof: Let $\epsilon > 0$ be given. We want to show that $\{n \in \mathbb{N} : \ |x_n - l| < \epsilon\}$ is infinite. To this end let $K = \mathbb{Q} \cap (l - \epsilon, \ l + \epsilon)$. We know that $K \neq \phi$ by density of $\mathbb{Q}$ in $\mathbb{R}$. Suppose K were finite. Then it would have a smallest element α, which is not possible since $(l - \epsilon, \alpha)$ would have to contain at least one rational number r, by virtue of density of $\mathbb{Q}$ in $\mathbb{R}$. This would contradict minimality of α. Since K is an infinite subset of the range of g and g is a bijection we conclude that

$$\{n \in \mathbb{N} : \ |x_n - l| < \epsilon\} = \{n \in \mathbb{N} : \ g(n) \in K\}$$

is infinite.