Practice Problems

Part I (Short Answer)

1. Consider the real sequence defined by

$$x_n = \left(\frac{(-1)^n + 1}{2}\right)^n + (-1)^n + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Find $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

2. Find the interior and closure of S if

$$S = \{-1\} \cup \{x \in \mathbb{Q} : 0 < x < 1\} \cup (1, 2].$$

3. Give an example of an infinite set $S \subset \mathbb{R}$ such that every subset of S is closed.

4. Determine whether or not f is uniformly continuous on S.

   (a) $S = \mathbb{R}, \quad f(x) = \dfrac{\sin(x^3)}{1 + x^2} \quad \text{for all } x \in S.$

   (b) $S = (0, 1), \quad f(x) = \dfrac{1}{x} \quad \text{for all } x \in S.$

   (c) $S = [0, 1], \quad f(x) = \dfrac{x^4}{1 + x} \quad \text{for all } x \in S.$

5. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that f is differentiable at exactly one point.

6. Give an example of a countably infinite collection $\{T_n : n \in \mathbb{N}\}$ of subsets of $\mathbb{R}$ such that

$$\bigcup_{n=1}^{\infty} cl(T_n) \neq cl\left(\bigcup_{n=1}^{\infty} T_n\right).$$

7. Determine whether or not $\{f_n\}_{n=1}^{\infty}$ converges uniformly on S.

   (a) $S = [0, \infty), \quad f_n(x) = \sqrt{\dfrac{x}{n}} \quad \text{for all } x \in S,\ n \in \mathbb{N}.$

   (b) $S = (0, 1), \quad f_n(x) = \sin^2\left(\dfrac{x}{n}\right) \quad \text{for all } x \in S,\ n \in \mathbb{N}.$

   (c) $S = (1, \infty), \quad f_n(x) = \dfrac{n}{n + x} \quad \text{for all } x \in S,\ n \in \mathbb{N}.$

8. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and T is a closed subset of $\mathbb{R}$, does it follow that $f[T]$ is closed. (Recall that $f[T] = \{f(x) : x \in T\}$.) Explain.

9. If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $f(0) = f(1) = f(2) = 0$, can we conclude that there exists $z \in (0, 2)$ with $f''(z) = 0$? Explain.

10. Give an example of two bounded sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that

$$\liminf_{n\to\infty}(x_n + y_n) > \left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right)$$

and

$$\limsup_{n\to\infty}(x_n + y_n) < \left(\limsup_{n\to\infty} x_n\right) + \left(\limsup_{n\to\infty} y_n\right).$$

Part II (Give Complete Proofs.)

1. Let S, T be subsets of $\mathbb{R}$. Show that

$$cl(S \cup T) = (cl(S)) \cup (cl(T)).$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be given and put $S = \{x \in \mathbb{R} : f(x) = 0\}$. Assume that f is differentiable on $\mathbb{R}$ and that $f'(x) = 0$ for all $x \in S$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = |f(x)|$ for all $x \in \mathbb{R}$. Show that g is differentiable on $\mathbb{R}$.

3. Let $S \subset \mathbb{R}$, $M, \alpha > 0$, and $f, g : S \to \mathbb{R}$ be given. Assume that

$$|f(x)| \leq M, \ |g(x)| \geq \alpha \quad \forall x \in S$$

and that f, g are uniformly continuous on S. Define $F : S \to \mathbb{R}$ by $F(x) = \dfrac{f(x)}{g(x)}$ $\forall x \in S$. Show that F is uniformly continuous on S.

4. Assume that $g : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $a_n \to 0$ as $n \to \infty$. Define the sequence $\{f_n\}_{n=1}^{\infty}$ of functions on $\mathbb{R}$ by

$$f_n(x) = g(x + a_n) \quad \text{for all } x \in \mathbb{R},\ n \in \mathbb{N}.$$

Show that $f_n \to g$ uniformly on $\mathbb{R}$ as $n \to \infty$.

5. Use the definition of limit to show that the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = \frac{3n^2}{2n^2-1}$ for all $n \in \mathbb{N}$ is convergent.

6. Let $f : [0, 1] \to \mathbb{R}$ be given and assume that

$$|f(x) - f(y)| \leq 6|x - y| \quad \forall x, y \in [0, 1].$$

Show that $f \in \mathcal{R}[a, b]$ **without** using the result which asserts that continuous functions on $[a, b]$ are integrable.