21-355 **Principles of Real Analysis I** **Fall 2004**

Assignment 7
Due on Friday, December 10

Let $a, b \in \mathbb{R}$ with $a < b$ be given.

1. For each $r \in \mathbb{Q}\backslash\{0\}$ there are unique integers $p(r)$ and $q(r)$ such that $q(r) > 0$, $p(r)$ and $q(r)$ have no common divisors, and $r = \dfrac{p(r)}{q(r)}$. Put $q(0) = 1$. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{for all } x \in [0,1]\backslash\mathbb{Q} \\ \dfrac{1}{q(x)} & \text{for all } x \in [0,1] \cap \mathbb{Q}. \end{cases}$$

Show that $f \in \mathcal{R}[0, 1]$.

2. Give an example of $f \in \mathcal{R}[0, 1]$ and an increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi \circ f \notin \mathcal{R}[0, 1]$.

3.* Let $g : \mathbb{R} \to \mathbb{R}$, $\alpha, \beta, r \in \mathbb{R}$ with $\beta > \alpha > r$ be given. Assume that g is strictly increasing and that $g(r) = 0$. Consider the sequence defined recursively by $x_1 = \beta, x_2 = \alpha$,

$$x_{n+2} = x_{n+1} - g(x_{n+1}) \left[\frac{x_{n+1} - x_n}{g(x_{n+1}) - g(x_n)} \right]$$

for all $n \in \mathbb{N}$. Formulate and prove a theorem ensuring that $x_n \to r$ as $n \to \infty$.

4.* Let $g : [a, b] \to \mathbb{R}$ be given and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions such that $f_n \in \mathcal{R}[a, b]$ for every $n \in \mathbb{N}$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence such that $\alpha_n > 0$ for every $n \in \mathbb{N}$ and $\alpha_n \to 0$ as $n \to \infty$. Assume that

$$|f_n(x) - g(x)| < \alpha_n \quad \forall n \in \mathbb{N},\ x \in [a, b].$$

Show that $g \in \mathcal{R}[a, b]$ and that

$$\int_a^b g = \lim_{n\to\infty} \int_a^b f_n.$$

5.* Let $f \in \mathcal{R}[a, b]$ be given and define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)dt \quad \forall x \in [a, b].$$

Show that F is uniformly continuous on $[a, b]$.

6.* Let $f, g \in \mathcal{R}[a, b]$ be given. Show that

$$\int_a^b fg \leq \left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}.$$

[Suggestion: Study the function $H : \mathbb{R} \to \mathbb{R}$ defined by

$$H(\lambda) = \int_a^b (f(x) - \lambda g(x))^2 dx \quad \forall \lambda \in \mathbb{R}.$$

Notice that $H(\lambda) \geq 0 \quad \forall \lambda \in \mathbb{R}$ and make a "magic choice" for λ.

*Problems marked with an asterisk should be written up and handed in.