Assignment 5

Due on Wednesday, November 17

Definition: Let $a, L \in \mathbb{R}$ and $g : (a, \infty) \to \mathbb{R}$ be given. We say that $g(x) \to L$ as $x \to \infty$ if $\forall \epsilon > 0$, $\exists M \geq a$ such that $|g(x) - L| < \epsilon$ for all $x > M$.

1.* For each $r \in \mathbb{Q} \backslash \{0\}$ there are unique integers $p(r)$, $q(r)$ with $q(r) > 0$ such that $p(r)$ and $q(r)$ have no common divisors and $r = \dfrac{p(r)}{q(r)}$. (This is called the reduced-fraction expansion for r.) Put $q(0) = 1$. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \forall x \in [0,1] \backslash \mathbb{Q} \\ \dfrac{1}{q(x)} & \forall x \in [0,1] \cap \mathbb{Q}. \end{cases}$$

Where (if anywhere) is f continuous?

2. Let $f : \mathbb{R} \to \mathbb{R}$, $K > 0$, and $\alpha > 1$ be given. Show that if $|f(x) - f(y)| \leq K|x - y|^\alpha$ for all $x, y \in \mathbb{R}$ then f is constant on $\mathbb{R}$.

3. Assume that $f : (0, \infty) \to \mathbb{R}$ is differentiable and that $f'(x) \to 0$ as $x \to \infty$. Show that $f(x + 1) - f(x) \to 0$ as $x \to \infty$.

4.* Let S be a subset of $\mathbb{R}$ and assume that $f, g : S \to \mathbb{R}$ are uniformly continuous on S and define $F : S \to \mathbb{R}$ by

$$F(x) = f(x)g(x) \quad \forall x \in S.$$

(a) Show that if f and g are bounded on S then F is uniformly continuous on S.

(b) What is the situation regarding uniform continuity of F if f is bounded but g is not?

5. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $f(x) \in \mathbb{Q}$ for all $x \in \mathbb{R}$. What can you conclude about f?

6.* Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and let $a, b, \alpha \in \mathbb{R}$ be given with $a < b$ and $f'(a) < \alpha < f'(b)$. Show that there exist $c \in (a, b)$ with $f'(c) = \alpha$.

7.* Let $f : \mathbb{R} \to \mathbb{R}$ be given. Show that if f is differentiable on $\mathbb{R}$ and f' is bounded on $\mathbb{R}$ then f is uniformly continuous on $\mathbb{R}$.

*Problems marked with an asterisk should be written up and handed in.