VII. Sequences of Functions

In Section II, we studied sequences of real numbers. It is very useful to consider extensions of this concept. More generally, a sequence is a function $F : \mathbb{N} \to \mathcal{S}$, where $\mathcal{S}$ is some set (possibly much more complicated than $\mathbb{R}$). For each $n \in \mathbb{N}$ the function value $F(n)$ is called the nth term of the sequence. We shall adopt the customary practice of writing F_n in place of $F(n)$ and denote the sequence by $\{F_n\}_{n=1}^{\infty}$. In order to talk about convergence of such a sequence, we need to know something about the structure of the set $\mathcal{S}$. In this section, we consider the case when $\mathcal{S}$ is the set of real-valued functions with domain S, where S is a given subset of $\mathbb{R}$. We refer to such sequences as sequences of functions, or sequences of functions on S. We shall consider two basic types of convergence for sequences of functions: *pointwise convergence* and *uniform convergence*. By a subsequence of $\{F_n\}_{n=1}^{\infty}$, we mean a sequence of the form $\{F_{n_k}\}_{k=1}^{\infty}$, where $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers.

A. Some Definitions

Let S be a subset of $\mathbb{R}$, $g : S \to \mathbb{R}$ and $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on S.

Definition 1: We say that $f_n \to g$ *pointwise* on S as $n \to \infty$ provided that

$$\forall x \in S, \quad f_n(x) \to g(x) \quad \text{as } n \to \infty.$$

We say that $\{f_n\}_{n=1}^{\infty}$ *converges pointwise* on S if there is a function $g_* : S \to \mathbb{R}$ such that $f_n \to g_*$ pointwise on S as $n \to \infty$.

Definition 2: We say that $f_n \to g$ *uniformly* on S as $n \to \infty$ provided that $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that $|f_n(x) - g(x)| < \epsilon \quad$ for all $x \in S,\ n \in \mathbb{N},\ n \geq N$. We say that $\{f_n\}_{n=1}^{\infty}$ *converges uniformly* on S if there is a function $g_* : S \to \mathbb{R}$ such that $f_n \to g_*$ uniformly on S as $n \to \infty$.

Definition 3: We say that $\{f_n\}_{n=1}^{\infty}$ is *uniformly Cauchy* on S provided that $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that $|f_m(x) - f_n(x)| < \epsilon$ for all $x \in S$ and all $m, n \in \mathbb{N}$ with $m, n \geq N$.

Definition 4: We say that $\{f_n\}_{n=1}^{\infty}$ is *uniformly equicontinuous* on S provided that $\forall \epsilon > 0,\ \exists \delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $x, y \in S$ with $|x - y| < \delta$ and all $n \in \mathbb{N}$.

B. Some Key Results

Let $S \subset \mathbb{R},\ y \in S,\ g : S \to \mathbb{R}$ and a sequence $\{f_n\}_{n=1}^{\infty}$ of functions on S be given.

VII.1 Proposition: $f_n \to g$ uniformly on S as $n \to \infty$ if and only if there is a real sequence $\{a_n\}_{n=1}^{\infty}$ (with $a_n \geq 0$ for all $n \in \mathbb{N}$) such that $a_n \to 0$ as $n \to \infty$ and $|f_n(x) - g(x)| \leq a_n$ for all $x \in S$, $n \in \mathbb{N}$.

VII.2 Theorem: Let $a, b \in \mathbb{R}$ with $a < b$ be given. Assume that $f_n \in \mathcal{R}[a, b]$ for every $n \in \mathbb{N}$ and let $g : [a, b] \to \mathbb{R}$ be given. Assume that $f_n \to g$ uniformly on $[a, b]$ as $n \to \infty$. Then $g \in \mathcal{R}[a, b]$ and

$$\int_a^b g = \lim_{n\to\infty} \int_a^b f_n.$$

VII.3 Theorem: Assume that f_n is continuous at y for every $n \in \mathbb{N}$ and that $f_n \to g$ uniformly on S as $n \to \infty$. Then g is continuous at y.

VII.4 Theorem: Assume that f_n is uniformly continuous on S for every $n \in \mathbb{N}$ and that $f_n \to g$ uniformly on S as $n \to \infty$. Then g is uniformly continuous on S.

VII.5 Theorem: $\{f_n\}_{n=1}^{\infty}$ converges uniformly on S if and only if $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy on S.

VII.6 Lemma: Assume that $\{f_n\}_{n=1}^{\infty}$ is uniformly equicontinuous on S and that $f_n \to g$ pointwise on S as $n \to \infty$. Then g is uniformly continuous on S.

VII.7 Lemma: Assume that S is compact and that f_n is continuous for every $n \in \mathbb{N}$. If $\{f_n\}_{n=1}^{\infty}$ converges uniformly on S then $\{f_n\}_{n=1}^{\infty}$ is uniformly equicontinuous on S.

VII.8 Lemma: Assume that S is compact, $\{f_n\}_{n=1}^{\infty}$ is uniformly equicontinuous on S and that $f_n \to g$ pointwise on S as $n \to \infty$. Then $f_n \to g$ uniformly on S as $n \to \infty$.

VII.9 Ascoli-Arzela Theorem: Assume that S is compact, $\{f_n\}_{n=1}^{\infty}$ is uniformly equicontinuous on S, and that $\forall x \in S$ the real sequence $\{f_n(x)\}_{n=1}^{\infty}$ is bounded. Then there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ that converges uniformly on S.

VII.10 Weierstrass Approximation Theorem: Assume that S is compact and that g is continuous. Then there is a sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials such that $P_n \to g$ uniformly on S as $n \to \infty$.

C. Some Remarks

VII.11 Remark: If $f_n \to g$ uniformly on S as $n \to \infty$ then $f_n \to g$ pointwise on S as $n \to \infty$. The converse implication is false.

VII.12 Remark: If $\{f_n\}_{n=1}^{\infty}$ is uniformly equicontinuous on S, then for each $n \in \mathbb{N}$, f_n is uniformly continuous on S.

VII.13 Remark:

a) We say that $\{f_n\}_{n=1}^{\infty}$ is equicontinuous at y provided that $\forall \epsilon > 0,\ \exists \delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $x \in S$ with $|x - y| < \delta$ and all $n \in \mathbb{N}$.

b) If $\{f_n\}_{n=1}^{\infty}$ is equicontinuous at each $y \in S$ and S is compact then $\{f_n\}_{n=1}^{\infty}$ is uniformly equicontinuous on S. (The proof is virtually identical to the proof of Theorem IV.8)

c) If $\{f_n\}_{n=1}^{\infty}$ is equicontinuous at y and $f_n \to g$ pointwise on S then g is continuous at y. (The proof is almost identical to the proof of Lemma V.6.)

D. Some Proofs

Proof of Theorem VII.3: Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$(1) \qquad |f_n(x) - g(x)| < \frac{\epsilon}{3} \quad \forall x \in S,\ n \in \mathbb{N},\ n \geq N.$$

Then choose $\delta > 0$ such that

$$(2) \qquad |f_N(x) - f_N(y)| < \frac{\epsilon}{3} \quad \forall x \in S \cap B_\delta(y).$$

For all $x \in S$ with $|x - y| < \delta$ we have

$$|g(x) - g(y)| \leq |g(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - g(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

by virtue of (1) and (2). ■

Proof of Theorem VII.4: Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$(3) \qquad |f_n(x) - g(x)| < \frac{\epsilon}{3} \quad \forall x \in S,\ n \in \mathbb{N},\ n \geq N.$$

Then choose $\delta > 0$ such that

$$(4) \qquad |f_N(x) - f_N(y)| < \frac{\epsilon}{3} \quad \forall x, y \in S,\ |x - y| < \delta.$$

For all $x, y \in S$ with $|x - y| < \delta$ we have

$$|g(x) - g(y)| \leq |g(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - g(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

by virtue of (3) and (4). ■

Proof of Theorem VII.5 Assume first that $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent of S. Choose $g_* : S \to \mathbb{R}$ such that $f_n \to g_*$ uniformly on S as $n \to \infty$. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$|f_n(x) - g_*(x)| < \frac{\epsilon}{2} \quad \forall x \in S,\ n \in \mathbb{N},\ n \geq N. \tag{5}$$

Then for all $m, n \in \mathbb{N}$ with $m, n \geq N$ and all $x \in S$, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - g_*(x)| + |g_*(x) - f_m(x)| < \epsilon/2 + \epsilon/2$$

by virtue of (5).

Assume now that $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy. Then for each $x \in S$, the real sequence $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence and consequently is convergent. Define $g : S \to \mathbb{R}$ by

$$g(x) = \lim_{n\to\infty} f_n(x) \quad \forall x \in S.$$

We shall show that $f_n \to g$ uniformly on S as $n \to \infty$. Let $\epsilon > 0$ be given. We choose $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2} \quad \forall x \in S,\ m, n \in \mathbb{N},\ m, n \geq N. \tag{6}$$

For each $x \in S$, we choose $N_x \in \mathbb{N}$ such that

$$|f_n(x) - g(x)| < \epsilon/2 \quad \forall n \in \mathbb{N},\ n \geq N_x, \tag{7}$$

and we put $N_x^* = \max\{N, N_x\}$. Then, for all $x \in S$ and all $n \in \mathbb{N}$ with $n \geq N$ we have

$$|f_n(x) - g(x)| \leq |f_n(x) - f_{N_x^*}(x)| + |f_{N_x^*}(x) - g(x)| < \epsilon/2 + \epsilon/2$$

by virtue of (6) and (7).

Proof of Lemma VII.6: Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \frac{\epsilon}{2} \quad \forall n \in \mathbb{N},\ x, y \in S,\ |x - y| < \delta. \tag{8}$$

Let $x, y \in S$ with $|x - y| < \delta$ be given. Then since $f_n(x) \to g(x)$ and $f_n(y) \to g(y)$ as $n \to \infty$, we may let $n \to \infty$ in (8) and deduce that $|g(x) - g(y)| \leq \frac{\epsilon}{2} < \epsilon$. ■

Proof of Lemma VII.8: Assume that $S \neq \phi$. Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \frac{\epsilon}{3} \quad \forall n \in \mathbb{N},\ x, y \in S,\ |x - y| < \delta. \tag{9}$$

Since $f_n \to g$ pointwise on S as $n \to \infty$, we may let $n \to \infty$ to find that

$$(10) \qquad |g(x) - g(y)| \leq \frac{\epsilon}{3} \quad \forall x, y \in S,\ |x - y| < \delta.$$

The collection $\{B_\delta(x) : x \in S\}$ of open sets covers S. Since S is compact and nonempty we may choose $\{x_1, x_2, \ldots, x_m\} \subset S$ such that $\{B_\delta(x_i) : i = 1, 2, \ldots m\}$ covers S. For each $i \in \{1, 2, \ldots, m\}$ choose $N_i \in \mathbb{N}$ such that

$$(11) \qquad |f_n(x_i) - g(x_i)| < \frac{\epsilon}{3} \quad \forall n \in \mathbb{N},\ n \geq N_i$$

and put $N = \max\{N_i : i = 1, 2, \ldots, m\}$. I claim that

$$|f_n(x) - g(x)| < \epsilon \quad \forall x \in S,\ n \in \mathbb{N},\ n \geq N.$$

To verify the claim, let $x \in S$ be given. Choose $i \in \{1, 2, \ldots, m\}$ such that $x \in B_\delta(x_i)$. Then we have

$$|f_n(x) - g(x)| = |f_n(x) - f_n(x_i)| + |f_n(x_i) - g(x_i)| + |g(x_i) - g(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

by virtue of (9), (10), and (11).

Sketch of Proof of Theorem VII.9:

Step 1: Construct a countable set $T \subset S$ with $cl(T) = S$.

Step 2: For every $x \in T$, the real sequence $\{f_n(x)\}_{n=1}^{\infty}$ is bounded. By the Bolzano-Weiersrtrass Theorem and a standard diagonalization argument one can construct a subsequence of $\{f_{n_k}\}_{k=1}^{\infty}$ such that $\{f_{n_k}(x)\}_{k=1}^{\infty}$ converges for each $x \in T$.

Step 3: Using uniform equicontinuity and the fact that $cl(T) = S$ one can show that $f_n \to g$ pointwise on S as $n \to \infty$.

Step 4: It follows from Lemma V.8 that $\{f_{n_k}(x)\}_{k=1}^{\infty}$ converges uniformly on S.