III. Topology of $\mathbb{R}$

A major goal of this course is to understand the behavior of real-valued functions defined on interesting subsets of $\mathbb{R}$. In order to give an efficient treatment of such functions it is useful to identify and study certain classes of subsets of $\mathbb{R}$. Two very important kinds of sets encountered in basic calculus are *open intervals* and *closed intervals*. An important generalization of open interval is provided by the notation of an *open set*; similarly an important generalization of closed interval is provided by the notion of a *closed set*. We shall show that a set $S \subset \mathbb{R}$ is closed if and only if its complement $S^c = \{x \in \mathbb{R} : x \notin S\}$ is open. Consequently, each statement that we prove about open sets leads immediately to a result about closed sets, and vice versa.

The branch of mathematics called *Topology* is concerned with the study of situations in which the notion of open set is fundamental. Our treatment of topology will be limited to open subsets of $\mathbb{R}$. However, I will try to present the basic definitions and results in such a way that extensions to more general frameworks will be as straightforward as possible.

Our definitions will be based on the notions of *ball* and *punctured ball*. For each $\delta > 0$ and $x_0 \in \mathbb{R}$, put

$$B_\delta(x_0) = \{x \in \mathbb{R} : |x - x_0| < \delta\},$$

$$B^*_\delta(x_0) = B_\delta(x_0) \backslash \{x_0\}.$$

We refer to $B_\delta(x_0)$ as the ball of radius δ centered at x_0 and to $B^*_\delta(x_0)$ as the punctured ball of radius δ centered at x_0.

A. Definitions

Let S be a subset of $\mathbb{R}$.

Definition 1: A point $x_0 \in \mathbb{R}$ is said to be an *interior point* of S if $\exists \delta > 0$ such that $B_\delta(x_0) \subset S$. The set of all *interior* points of S is called the interior of S and is denoted by $\mathrm{int}(S)$.

Definition 2: A point $x_0 \in \mathbb{R}$ is said to be a *point of closure* of S if $\forall \delta > 0,\ B_\delta(x_0) \cap S \neq \emptyset$. The set of all points of closure of S is called the *closure* of S and is denoted by $cl(S)$.

Definition 3: A point $x_0 \in \mathbb{R}$ is said to be an *accumulation point* or a *limit point* of S if $\forall \delta > 0,\ B^*_\delta(x_0) \cap S \neq \emptyset$. The set of all accumulation points of S is denoted by $acc(S)$.

Definition 4: We say that S is *open* if $S = \mathrm{int}(S)$.

Definition 5: We say that S is *closed* if $S = cl(S)$.

Definition 6: We say that S is *dense* (in $\mathbb{R}$) if $cl(S) = \mathbb{R}$.

Definition 7: Let $\mathcal{C}$ be a collection of subsets of $\mathbb{R}$. We say that $\mathcal{C}$ *covers* S if $S \subset \cup\mathcal{C}$.

Definition 8: We say that S is *compact* if every collection of open sets that covers S has a finite subcollection that also covers S.

Definition 9: We say that S is an *interval* provided that $tx + (1-t)y \in S$ for all $x, y \in S$ and $t \in \mathbb{R}$ with $0 \le t \le 1$.

Definition 10: We say that S is

(a) of class $\mathcal{G}_\delta$ if S can be expressed as the intersection of a countable collection of open sets.

(b) of class $\mathcal{F}_\sigma$ is S can be expressed as the union of a countable collection of closed sets.

B. Some Key Results

III.1 Proposition: Let $a, b \in \mathbb{R}$ with $a < b$ be given, Then the intervals $(-\infty, b)$, (a, b), and (a, ∞) are open. The intervals $(-\infty, b]$, $[a, b]$, and $[a, \infty)$ are closed.

III.2 Proposition: Let S be a subset of $\mathbb{R}$. Then $\text{int}(S)$ is open and $cl(S)$ is closed.

III.3 Theorem: Let S be a subset of $\mathbb{R}$. Then S is closed if and only if S^c is open.

III.4 Theorem:

(i) The union of any collection of open sets is open. (ii) The intersection of any finite collection of open sets is open. (iii) The intersection of any collection of closed sets is closed. (iv) The union of any finite collection of closed sets is closed.

III.5 Proposition: Let S be a subset of $\mathbb{R}$ and $\ell \in \mathbb{R}$ be given. Then $\ell \in cl(S)$ if and only if there is a sequence $\{x_n\}_{n=1}^\infty$ with $x_n \in S$ for every $n \in \mathbb{N}$ and $x_n \to \ell$ as $n \to \infty$.

III.6 Proposition: Let S be a subset of $\mathbb{R}$. Then S is closed if and only if for every convergent sequence $\{x_n\}_{n=1}^\infty$ with $x_n \in S$ for all $n \in \mathbb{N}$ we have $(\lim_{n\to\infty} x_n) \in S$.

III.7 Proposition: Let S be a subset of $\mathbb{R}$. Then $acc(S)$ is closed and $cl(S) = S \cup acc(S)$.

III.8 Heine-Borel Theorem: Let S be a subset of $\mathbb{R}$. Then S is compact if and only if S is closed and bounded.

III.9 Theorem: Let S be an open subset of $\mathbb{R}$. Then there is a collection $\{I_i : i \in \mathbb{N}\}$ of open intervals such that $I_i \cap I_j = \emptyset$ for all $i, j \in \mathbb{N}$ with $i \neq j$ and $S = \cup_{i=1}^{\infty} I_i$.

III.10 Theorem: Let S be a subset of $\mathbb{R}$. If S is both open and closed then $S = \mathbb{R}$ or $S = \emptyset$.

C. Some Remarks

III.11 Remark: For every set $S \subset \mathbb{R}$ we have $\text{int}(S) \subset S \subset cl(S)$. To show that S is open it suffices to verify that $S \subset \text{int}(S)$. To show that S is closed it suffices to verify that $cl(S) \subset S$.

III.12 Remark: Since $(S^c)^c = S$, it follows from Theorem III.3 that S is open if and only if S^c is closed.

III.13 Remark: Every finite subset of $\mathbb{R}$ is compact.

III.14 Remark: It follows from Proposition III.6, Theorem III.8, and the Bolzano-Weierstrass Theorem that a set $S \subset \mathbb{R}$ is compact if and only if every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S \quad \forall n \in \mathbb{N}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with $(\lim_{k\to\infty} x_{n_k}) \in S$.

III.15 Remark: It is not difficult to show that if $I \subset \mathbb{R}$ is an interval, then I must have one of the forms below:

$$I = \emptyset,\ I = \mathbb{R},\ I = \{a\},\ I = (-\infty, b),$$

$$I = (-\infty, b],\ I = (a, \infty),\ I = [a, \infty),$$

$$I = (a, b),\ I = (a, b],\ I = [a, b),$$

$$I = [a, b].$$

Conversely, each of the above sets is an interval.

D. Some Proofs

Proof of III.3: Assume first that S is closed. We shall show that S^c is open. Let $x_0 \in S^c$ be given. We want to show that $x_0 \in \text{int}(S^c)$. Since S is closed and $x_0 \notin S$, we know that $x_0 \notin cl(S)$. Therefore we may choose $\delta > 0$ such that $B_\delta(x_0) \cap S = \emptyset$.

It follows that $B_\delta(x_0) \subset S^c$ and $x_0 \in \text{int}(S^c)$.

Assume now that S^c is open. We shall show that S is closed. For this purpose let $x_0 \in cl(S)$ be given. We want to show that $x_0 \in S$. For every $\delta > 0$ we have $B_\delta(x_0) \cap S \neq \emptyset$. It follows that for every $\delta > 0$, $B_\delta(x_0) \not\subset S^c$. We conclude that $x_0 \notin \text{int}(S^c)$. Since S^c is open, it follows that $x_0 \notin S^c$, i.e. $x_0 \in S$. ■

Proof of III.4. (i) Let $\{S_i : i \in I\}$ be a collection of open sets and put $U = \bigcup_{i \in I} S_i$. Let $x_0 \in U$ be given. Then we may choose $j \in I$ such that $x_0 \in S_j$. Since S_j is open we may choose $\delta > 0$ such that $B_\delta(x_0) \subset S_j$. It follows that $B_\delta(x_0) \subset U$ and $x_0 \in \text{int}(U)$. We conclude that U is open.

(ii) Let F be a finite set and $\{S_i : i \in F\}$ be a collection of open sets. Put $V = \bigcap_{i \in F} S_i$. If $F = \emptyset$ then $V = \mathbb{R}$ and hence V is open. Assume that $F \neq \emptyset$ and let $x_0 \in V$ be given. Then $x_0 \in S_i = \text{int}(S_i)$ for every $i \in F$. Consequently, for each $i \in F$ we may choose $\delta_i > 0$ such that $B_{\delta_i}(x_0) \subset S_i$. Since $\{\delta_i : i \in F\}$ is nonempty and finite it has a smallest element. Let $\delta = \min\{\delta_i : i \in F\}$ and note that $\delta > 0$. Then for all $i \in F$ we have

$$B_\delta(x_0) \subset B_{\delta_i}(x_o) \subset S_i. \tag{1}$$

We conclude that $B_\delta(x_0) \subset V$. It follows that $x_0 \in \text{int}(V)$ and V is open.

(iii) and (iv) follow by applying Theorem III.3 and DeMorgan's Laws to (i) and (ii). ■

Proof of III.5: Assume first that $l \in cl(S)$. Then for each $\delta > 0$ we have $B_\delta(l) \cap S \neq \emptyset$. For each $n \in \mathbb{N}$ we choose $x_n \in B_{\frac{1}{n}}(l) \cap S$. Notice that

$$l - \frac{1}{n} < x_n < l + \frac{1}{n} \quad \forall n \in \mathbb{N}. \tag{2}$$

The Squeeze Theorem implies that $x_n \to l$ as $n \to \infty$.

To prove the converse, let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \in S \quad \forall n \in \mathbb{N}$ and $x_n \to l$ as $n \to \infty$. Let $\delta > 0$ be given. Then we may choose $N \in \mathbb{N}$ such that

$$|x_n - l| < \delta \quad \forall n \in \mathbb{N},\ n \geq N, \text{ i.e.} \tag{3}$$

$$x_n \in B_\delta(l) \quad \forall n \in \mathbb{N},\ n \geq N. \tag{4}$$

In particular $x_N \in B_\delta(l)$. Since $x_N \in S$, we conclude that $B_\delta(l) \cap S \neq \emptyset$. Since $\delta > 0$ was arbitrary, it follows that $l \in cl(S)$. ■

Proof of Theorem III.8: The proof will be partioned into 4 lemmas.

Lemma 1: Let S be a compact subset of $\mathbb{R}$. Then S is bounded.

Proof of Lemma 1: The collection $\{B_1(x) : x \in S\}$ of open sets covers S. Since S is compact we may choose a finite set $F \subset S$ such that

$$S \subset \bigcup_{i \in F} B_1(i). \tag{5}$$

Let β be an upper bound for F and α be a lower bound for F. Then

$$\alpha - 1 < x < \beta + 1 \quad \forall i \in F,\ x \in B_1(i). \tag{6}$$

It follows from (5) and (6) that

$$\alpha - 1 < x < \beta + 1 \quad \forall x \in S \tag{7}$$

and S is bounded. ■

Lemma 2: Let S be a compact subset of $\mathbb{R}$. Then S is closed.

Proof of Lemma 2: We shall show that S^c is open. Let $x_0 \in S^c$ be given. For each $x \in S$, put

$$\delta_x = \frac{1}{3}|x - x_0| \tag{8}$$

and notice that $\delta_x > 0$. Now, for each $x \in S$, put

$$\mathcal{U}_x = B_{\delta_x}(x), \tag{9}$$

$$\mathcal{O}_x = B_{\delta_x}(x_0). \tag{10}$$

Observe that

$$\mathcal{U}_x \cap \mathcal{O}_x = \emptyset \quad \forall x \in S. \tag{11}$$

The collection $\{\mathcal{U}_x : x \in S\}$ of open sets covers S. Since S is compact we may choose a finite set $F \subset S$ such that

$$S \subset \bigcup_{x \in F} \mathcal{U}_x. \tag{12}$$

If $F = \emptyset$, then $S = \emptyset$ and $S^c = \mathbb{R}$ so we are done. Assume that $F \neq \emptyset$. Then $\{\delta_x : x \in F\}$ has a smallest element. Let $\delta = \min\{\delta_x : x \in F\}$ and note that $\delta > 0$. Now, let

(13) $$\mathcal{O} = \bigcap_{x \in F} \mathcal{O}_x = B_\delta(x_0).$$

Observe that

(14) $$B_\delta(x_0) \cap \mathcal{U}_x = \emptyset \quad \forall x \in F$$

by virtue of (11) and the fact that $\mathcal{O} \subset \mathcal{O}_x$ for every $x \in F$. We shall show that $B_\delta(x_0) \subset S^c$. It follows from (14) that

(15) $$B_\delta(x_0) \cap \big(\bigcup_{x \in F} \mathcal{U}_x\big) = \emptyset$$

which yields

(16) $$B_\delta(x_0) \cap S = \emptyset$$

by virtue of (12). We infer from (16) that $B_\delta(x_0) \subset S^c$ and consequently S^c is open. ■

Lemma 3: Let $a, b \in \mathbb{R}$ with $a < b$ be given. The interval $[a, b]$ is compact.

Proof of Lemma 3: Let $\mathcal{C}$ be a collection of open sets that covers $[a, b]$. We define E to be the set of all $x \in (a, b]$ such that the interval $[a, x]$ can be covered by a finite subcollection of $\mathcal{C}$. Our goal is to show that $b \in E$.

Claim 1: $E \neq \emptyset$.

Proof of Claim 1: Choose $\mathcal{O}_a \in \mathcal{C}$ such that $a \in \mathcal{O}_a$ and then choose $\delta_a > 0$ such that $B_{\delta_a}(a) \subset \mathcal{O}_a$. Let $x_a = \min\{b, a + \frac{1}{2}\delta_a\}$ and notice that $x_a \in (a, b]$ and

(17) $$[a, x_a] \subset \mathcal{O}_a.$$

It follows that $x_a \in E$ and $E \neq \emptyset$. //

Claim 2: E is bounded above.

Proof of Claim 2: b is an upper bound for E. //

Let $c = \sup(E)$. We shall show that $c = b$ and $c \in E$.

Claim 3: $c = b$.

Proof of Claim 3: Observe that $c \leq b$, since b is an upper bound for E. Observe also that $c > a$. Suppose that $c < b$. Choose $\mathcal{O}_c \in \mathcal{C}$ such that $c \in \mathcal{O}_c$ and then choose $\delta_c > 0$ such that $B_{\delta_c}(c) \subset \mathcal{O}_c$. Since $c - \delta_c$ is not an upper bound for E, we may choose $\gamma \in (c - \delta_c, c]$ with $\gamma \in E$. Let $\mathcal{F}$ be a finite subcollection of $\mathcal{C}$ that covers $[a, \gamma]$. Then the finite subcollection $\mathcal{F} \cup \{\mathcal{O}_c\}$ of $\mathcal{C}$ covers $[a, c + \delta_c)$. Since $c < b$, we can produce an element of E that is strictly greater than c. [$x_c = \min \{b, c + \frac{1}{2}\delta_c\}$ will do.] This contradicts the fact that c is an upper bound for E. //

Claim 4: $b \in E$

Proof of Claim 4: Choose $\mathcal{U} \in \mathcal{C}$ such that $b \in \mathcal{U}$ and then choose $\delta > 0$ such that $B_\delta(b) \subset \mathcal{U}$. Since $b - \delta$ is not an upper bound for E we may choose $\Gamma \in (b - \delta, b]$ with $\Gamma \in E$. Let $\mathcal{G}$ be a finite subcollection of $\mathcal{C}$ that covers $[a, \Gamma]$. Then the finite subcollection $\mathcal{G} \cup \{\mathcal{U}\}$ of $\mathcal{C}$ covers $[a, b + \delta)$; in particular this finite subcollection covers $[a, b]$. It follows that $b \in E$. //

This completes the proof of Lemma 3. ■

Lemma 4: Let S be a compact subset of $\mathbb{R}$ and K be a closed subset of S. Then K is compact.

Proof of Lemma 4: Let $\mathcal{C}$ be a collection of open sets that covers K. Then $\mathcal{C} \cup \{K^c\}$ is a collection of open sets that covers $\mathbb{R}$ and hence also covers S. Since S is compact we may choose a finite subcollection $\mathcal{F}$ of $\mathcal{C} \cup \{K^c\}$ that covers S. Then $\mathcal{F} \setminus \{K^c\}$ is a finite subcollection of $\mathcal{C}$ that covers K and K is compact. ■

The Heine-Borel Theorem follows easily from Lemmas 1 through 4. If S is compact then S is closed and bounded by Lemmas 1 and 2. Suppose now that S is closed and bounded and choose $a, b \in \mathbb{R}$ with $a < b$ such that

$$a \leq x \leq b \quad \forall x \in S. \tag{18}$$

Then $[a, b]$ is compact by Lemma 3. Since S is a closed subset of $[a, b]$, we deduce from Lemma 4 that S is compact. ■