II. Sequences

By a *real sequence* we mean a function $x : \mathbb{N} \to \mathbb{R}$, i.e. a function whose domain is the set of natural numbers and whose values are real numbers. For each $n \in \mathbb{N}$ the function value $x(n)$ is called the nth *term* of the sequence. It is customary to write x_n in place of $x(n)$ and to denote the sequence by $\{x_n\}_{n=1}^{\infty}$. Although we will generally adopt the customary notation, it is important to bear in mind that a sequence is a function. Throughout this section we use the term sequence to mean real sequence. Most of our effort with sequences will be devoted to understanding how the terms x_n behave when the *index* n is large.

The central notion pertaining to sequences is that of a *limit*. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and $l \in \mathbb{R}$ be given. We say that l is a limit of $\{x_n\}_{=1}^{\infty}$ and we write $x_n \to l$ as $n \to \infty$ provided that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - l| < \epsilon$ for all $n \in \mathbb{N}$ with $n \geq N$. A sequence can have at most one limit. (See Proposition II.1.) Therefore, if $x_n \to l$ as $n \to \infty$, we refer to l as *the* limit of the sequence and we write $\lim\limits_{n\to\infty} x_n = l$.

A. Some Definitions

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence.

Definition 1: We say that $\{x_n\}_{n=1}^{\infty}$ is *convergent* if there exists $l \in \mathbb{R}$ such that $x_n \to l$ as $n \to \infty$.

Definition 2: We say that $\{x_n\}_{n=1}^{\infty}$ is
(i) *bounded below* if there exists $\alpha \in \mathbb{R}$ such that $x_n \geq \alpha$ for all $n \in \mathbb{N}$.
(ii) *bounded above* if there exists $\beta \in \mathbb{R}$ such that $x_n \leq \beta$ for all $n \in \mathbb{N}$.
(iii) *bounded* if there exists $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Definition 3: We say that $\{x_n\}_{n=1}^{\infty}$ is
(i) *increasing* if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.
(ii) *strictly increasing* if $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.
(iii) *decreasing* if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$.
(iv) *strictly decreasing* if $x_{n+1} < x_n$ for all $n \in \mathbb{N}$.
(v) *monotonic* if it is either increasing or decreasing.
(vi) *strictly monotonic* if it is either strictly increasing or strictly decreasing.

Definition 4: We say that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence provided that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_m - x_n| < \epsilon$ for all $m, n \in \mathbb{N}$ with $m, n \geq N$.

Definition 5: By a subsequence of $\{x_n\}_{n=1}^{\infty}$ we mean a sequence of the form $\{x_{n_k}\}_{k=1}^{\infty}$ where $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers.

Definition 6: Let $l \in \mathbb{R}$ be given. We say that l is a cluster point of $\{x_n\}_{n=1}^{\infty}$ provided that for every $\epsilon > 0$, $\{n \in \mathbb{N} : |x_n - l| < \epsilon\}$ is infinite.

Definition 7: Assume that $\{x_n\}_{n=1}^{\infty}$ is bounded. For each $n \in \mathbb{N}$ put

$$y_n = \inf\{x_k : k \in \mathbb{N},\ k \geq n\},$$

$$z_n = \sup\{x_k : k \in \mathbb{N},\ k \geq n\}.$$

Note that $\{y_n\}_{n=1}^{\infty}$ is increasing and bounded above and that $\{z_n\}_{n=1}^{\infty}$ is decreasing and bounded below. We define

$$\liminf_{n\to\infty}\ x_n = \lim_{n\to\infty} y_n \quad \text{and} \quad \limsup_{n\to\infty}\ x_n = \lim_{n\to\infty}\ z_n.$$

(Note that $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are convergent by virtue of Theorem II.6.)

B. Some Key Results

II.1 Proposition: A sequence can have at most one limit.

II.2 Proposition: Every convergent sequence is bounded.

II.3 Proposition: Let $\ell, L, \alpha \in \mathbb{R}$ be given and $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be sequences. Assume that $x_n \to \ell$ and $y_n \to L$ as $n \to \infty$. Then:

(i) $x_n + y_n \to \ell + L$ as $n \to \infty$;
(ii) $\alpha x_n \to \alpha\ell$ as $n \to \infty$;
(iii) $x_n y_n \to \ell L$ as $n \to \infty$;
(iv) If $x_n \neq 0$ for all $n \in \mathbb{N}$ and $\ell \neq 0$, we have $\frac{1}{x_n} \to \frac{1}{\ell}$ as $n \to \infty$.

II.4 Proposition: Let $\ell, L \in \mathbb{R}$ be given and $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be sequences. If $x_n \leq y_n$ for all $n \in \mathbb{N}$ and $x_n \to \ell$, $y_n \to L$ as $n \to \infty$ then $\ell \leq L$.

II.5 Squeeze Theorem: Let $\ell \in \mathbb{R}$ be given and $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ be sequences. Assume that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and that $x_n \to \ell$, $z_n \to \ell$ as $n \to \infty$. Then $y_n \to \ell$ as $n \to \infty$.

II.6 Theorem: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence.
(i) If $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded above then $\{x_n\}_{n=1}^{\infty}$ is convergent.
(ii) If $\{x_n\}_{n=1}^{\infty}$ is decreasing and bounded below then $\{x_n\}_{n=1}^{\infty}$ is convergent.

II.7 Proposition: Let $\ell \in \mathbb{R}$ be given and $\{x_n\}_{n=1}^{\infty}$ be a sequence. Then ℓ is a cluster point of $\{x_n\}_{n=1}^{\infty}$ if and only if there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to \ell$ as $k \to \infty$.

II.8 Proposition: Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences and $\alpha \in \mathbb{R}$ be given. Then:

(i) $\limsup_{n\to\infty}(x_n + y_n) \le \left(\limsup_{n\to\infty} x_n\right) + \left(\limsup_{n\to\infty} y_n\right)$;
(ii) $\liminf_{n\to\infty}(x_n + y_n) \ge \left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right)$;
(iii) If $\alpha \ge 0$ we have $\limsup_{n\to\infty}(\alpha x_n) = \alpha \limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty}(\alpha x_n) = \alpha \liminf_{n\to\infty} x_n$;
(iv) $\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n$ and $\liminf_{n\to\infty}(-x_n) = -\limsup_{n\to\infty} x_n$

II.9 Lemma: Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence and $l_s \in \mathbb{R}$ be given. Then $l_s = \limsup_{n\to\infty} x_n$ if and only if (i) and (ii) below hold.

(i) $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that $x_n < l_s + \epsilon$ for all $n \in \mathbb{N}$ with $n \ge N$.
(ii) $\forall \epsilon > 0,\ \forall N \in \mathbb{N},\ \exists n \in \mathbb{N}$ with $n \ge N$ such that $x_n > l_s - \epsilon$.

II.10 Proposition: Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. Then $\limsup_{n\to\infty} x_n$ is the largest cluster point of $\{x_n\}_{n=1}^{\infty}$ and $\liminf_{n\to\infty} x_n$ is the smallest cluster point of $\{x_n\}_{n=1}^{\infty}$.

II.11 Proposition: Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence and put $l_i = \liminf_{n\to\infty} x_n$ and $l_s = \limsup_{n\to\infty} x_n$. Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that

$$l_i - \epsilon < x_n < l_s + \epsilon$$

for all $n \in \mathbb{N}$ with $n \ge N$.

II.12 Proposition: Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. Then $\{x_n\}_{n=1}^{\infty}$ is convergent if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n.$$

II.13 Bolzano-Weierstrass Theorem: Every bounded sequence has a convergent subsequence.

II.14 Theorem (Cauchy's Criterion): A sequence is convergent if and only if it is a Cauchy sequence.

II.15 Lemma: Every sequence has a monotonic subsequence.

C. Some Remarks.

II.16 Remark: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Then $\{x_n\}_{n=1}^{\infty}$ is
(i) increasing if and only if $x_m \ge x_n$ for all $m, n \in \mathbb{N}$ with $m \ge n$.
(ii) strictly increasing if and only if $x_m > x_n$ for all $m, n \in \mathbb{N}$ with $m > n$.
(iii) decreasing if and only if $x_m \le x_n$ for all $m, n \in \mathbb{N}$ with $m \ge n$.
(iv) strictly decreasing if and only if $x_m < x_n$ for all $m, n \in \mathbb{N}$ with $m > n$.

II.17 Remark: Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers. Then $n_k \ge k$ for all $k \in \mathbb{N}$.

II.18 Remark: Let $\mathbb{K}$ be an infinite subset of $\mathbb{N}$. Then there is exactly one strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $\{n_k : k \in \mathbb{N}\} = \mathbb{K}$.

II.19 Remark: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and $l \in \mathbb{R}$ be given. Then l is a cluster point of $\{x_n\}$ if and only if for every $\epsilon > 0$ and every $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ with $n \geq N$ such that $|x_n - l| < \epsilon$.

D. Some Proofs.

Proof of II.1: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and let $l, L \in \mathbb{R}$ be given. Suppose that $x_n \to l$ as $n \to \infty$ and that $x_n \to L$ as $n \to \infty$. We shall show that $L = l$. Let $\epsilon > 0$ be given. Choose $N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - l| < \epsilon \quad \forall n \in \mathbb{N}, n \geq N_1, \tag{1}$$

$$|x_n - L| < \epsilon \quad \forall n \in \mathbb{N},\ n \geq N_2. \tag{2}$$

Put $N = \max\{N_1, N_2\}$ and notice that

$$|x_N - l| < \epsilon, \quad |x_N - L| < \epsilon. \tag{3}$$

Now we observe that

$$l - L = l - x_N + x_N - L \tag{4}$$

and consequently

$$|l - L| \leq |l - x_N| + |x_N - L| < \epsilon + \epsilon = 2\epsilon \tag{5}$$

by virtue of the triangle inequality and (3). Since $\epsilon > 0$ was arbitrary, it follows from (5) that $l - L = 0$. [Indeed, if $l - L \neq 0$ then we may put $\epsilon = \frac{1}{2}|l - L|$ in (5) which yields $|l - L| < |l - L|$ and this is impossible.] ■

Proof of II.2: Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence and put $l = \lim\limits_{n\to\infty} x_n$. Using the definition of limit with $\epsilon = 1$, we choose $N \in \mathbb{N}$ such that

$$|x_n - l| < 1 \quad \forall n \in \mathbb{N},\ n \geq N. \tag{6}$$

Let $S = \{|x_1|, |x_2|, \ldots |x_N|\}$. Since S is nonempty and finite, it has a largest element. Let $K = \max\ (S)$ and $M = \max\{1 + |l|, K\}$. Let $n \in \mathbb{N}$ be given. If $n \leq N$ then $|x_n| \in S$ so that

$$|x_n| \le K \le M. \tag{7}$$

If $n \ge N$, then we have

$$x_n = l + x_n - l \tag{8}$$

which yields

$$|x_n| \le |l| + |x_n - l| \le |l| + 1 \le M \tag{9}$$

by virtue of the triangle inequality, (6), and the definition of M. We conclude that $|x_n| \le M$ for all $n \in \mathbb{N}$, i.e. $\{x_n\}_{n=1}^{\infty}$ is bounded. ■

Proof of II.3 (i): Let $\epsilon > 0$ be given. Choose $N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - l| < \frac{\epsilon}{2} \quad \forall n \in \mathbb{N},\ n \ge N_1, \tag{10}$$

$$|y_n - L| < \frac{\epsilon}{2} \quad \forall n \in \mathbb{N},\ n \ge N_2 \tag{11}$$

and put $N = \max\{N_1, N_2\}$. Then for all $n \in \mathbb{N}$ with $n \ge N$ we have

$$\begin{aligned} &|x_n + y_n - (l + L)| = |(x_n - l) + (y_n - L)| \\ &\le |x_n - l| + |y_n - L| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \end{aligned} \tag{12}$$

by virtue of the triangle inequality and (10), (11). ■

Proof of II.3 (iii): Since $\{x_n\}_{n=1}^{\infty}$ is convergent we may choose $M > 0$ such that

(13). $$|x_n| \le M \quad \forall n \in \mathbb{N}.$$

Let $\epsilon > 0$ be given. Choose $N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - l| < \frac{\epsilon}{2(|L| + 1)} \quad \forall n \in \mathbb{N},\ n \ge N_1, \tag{14}$$

$$|y_n - L| < \frac{\epsilon}{2M} \quad \forall n \in \mathbb{N},\ n \ge N_2. \tag{15}$$

Put $N = \max\{N_1, N_2\}$. Then, for all $n \in \mathbb{N}$ with $n \geq N$ we have

$$
\begin{aligned}
&|x_n y_n - lL| = |x_n y_n - Lx_n + Lx_n - lL| \\
&= |x_n(y_n - L) + L(x_n - l)| \\
&\leq |x_n| \cdot |y_n - L| + |L| \cdot |x_n - l| \\
&< M\left(\frac{\epsilon}{2M}\right) + \frac{|L|\epsilon}{2(|L|+1)} < \epsilon
\end{aligned} \tag{16}
$$

by virtue of (13), (14), (15). ■

Proof of II.4: Assume that $x_n \leq y_n$ for all $n \in \mathbb{N}$ and that $x_n \to l$, $y_n \to L$ as $n \to \infty$. Put

$$z_n = y_n - x_n \quad \forall n \in \mathbb{N}, \tag{17}$$

$$\alpha = L - l \tag{18}$$

and notice that $z_n \geq 0$ for all $n \in \mathbb{N}$ and that $z_n \to \alpha$ as $n \to \infty$. We shall show that $\alpha \geq 0$, which yields $l \leq L$.

Suppose that $\alpha < 0$. Then we may choose $N \in \mathbb{N}$ such

$$|z_n - \alpha| < -\frac{\alpha}{2} \quad \forall n \in \mathbb{N},\ n \geq N, \text{ i.e.} \tag{19}$$

$$\frac{\alpha}{2} < z_n - \alpha < -\frac{\alpha}{2} \quad \forall n \in \mathbb{N},\ n \geq N. \tag{20}$$

It follows from (20) that

$$z_N < \frac{\alpha}{2} < 0 \tag{21}$$

and this is a contradiction (since $z_n \geq 0$ for all $n \in \mathbb{N}$). We therefore conclude that $\alpha \geq 0$ and hence that $l \leq L$. ■

Proof of II.6 (i): Assume that $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded above. Put $S = \{x_n : n \in \mathbb{N}\}$ and observe that S is nonempty and bounded above. Let

$$l = \sup(S). \tag{22}$$

We shall show that $x_n \to l$ as $n \to \infty$. Let $\epsilon > 0$ be given. Then $l - \epsilon$ is not an upper bound for S. We may therefore choose $N \in \mathbb{N}$ such that

$$x_N > l - \epsilon \tag{23}$$

Recall that

$$x_n \leq l \quad \forall n \in \mathbb{N}. \tag{24}$$

Since $\{x_n\}_{n=1}^{\infty}$ is increasing we deduce from (23) and (24) that

$$l - \epsilon < x_N \leq x_n \leq l \quad \forall n \in \mathbb{N},\ n \geq N. \tag{25}$$

It follows from (25) that

$$|x_n - l| < \epsilon \quad \forall n \in \mathbb{N},\ n \geq N. \qquad \blacksquare \tag{26}$$

Proof of II.9: For each $n \in \mathbb{N}$, put

$$T_n = \{x_k : k \in \mathbb{N}, k \geq n\}, \tag{27}$$

$$z_n = \sup(T_n). \tag{28}$$

Recall that $\{z_n\}_{n=1}^{\infty}$ is decreasing and that

$$\lim_{n\to\infty} z_n = \limsup_{n\to\infty}\ x_n. \tag{29}$$

Assume first that $l_s = \limsup\limits_{n\to\infty} x_n$. We shall show that (i) and (ii) hold. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$|z_n - l_s| < \epsilon \quad \forall n \in \mathbb{N},\ n \geq N. \tag{30}$$

Then, for all $n \in \mathbb{N}$ with $n \geq N$ we have

$$z_n - l_s < \epsilon, \quad \text{i.e.} \tag{31}$$

$$z_n < l_s + \epsilon, \tag{32}$$

which yields

$$x_n \leq z_n < l_s + \epsilon \tag{33}$$

and consequently (i) holds. To verify (ii), let $\epsilon > 0$ and $N \in \mathbb{N}$ be given. Since $\{z_n\}_{n=1}^{\infty}$ is decreasing and $z_n \to l_s$ as $n \to \infty$, we know that

$$z_n \geq l_s > l_s - \epsilon \quad \forall n \in \mathbb{N}. \tag{34}$$

It follows from (34) that $l - \epsilon$ is not an upper bound for T_N. We may therefore choose $y \in T_N$ with $y > l_s - \epsilon$. By the definition of T_N, $y = x_n$ for some $n \in \mathbb{N}$ with $n \geq N$.

Conversely, assume now that (i) and (ii) hold. We shall show that $l_s = \limsup_{n\to\infty} x_n$. Let $\epsilon > 0$ be given. It follows from (ii) that

$$z_n > l_s - \epsilon \quad \forall n \in \mathbb{N}. \tag{35}$$

Using (i), we choose $N \in \mathbb{N}$ such that

$$x_n < l_s + \frac{\epsilon}{2} \quad \forall n \in \mathbb{N},\ n \geq N. \tag{36}$$

It follows from (36) that

$$z_n \leq l_s + \frac{\epsilon}{2} < l_s + \epsilon. \tag{37}$$

Since $\{z_n\}_{n=1}^{\infty}$ is decreasing, (37) yields

$$z_n < l_s + \epsilon \quad \forall n \in \mathbb{N},\ n \geq N. \tag{38}$$

Combining (35) and (38) we arrive at

$$|z_n - l_s| < \epsilon \quad \forall n \in \mathbb{N},\ n \geq N. \tag{39}$$

We conclude that $z_n \to l_s$ as $n \to \infty$ and consequently $l_s = \limsup_{n\to\infty} x_n$. ∎

Proof of II.12: For each $n \in \mathbb{N}$, put

$$T_n = \{x_k : k \in \mathbb{N},\ k \geq n\}, \tag{40}$$

$$y_n = \inf(T_n), \tag{41}$$

$$z_n = \sup(T_n). \tag{42}$$

Let $l \in \mathbb{R}$ be given. Assume first that $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = l$. We shall show that $x_n \to l$ as $n \to \infty$. Observe that

$$y_n \le x_n \le z_n \quad \forall n \in \mathbb{N}. \tag{43}$$

Since $y_n \to l$ and $z_n \to l$ as $n \to \infty$, it follows from the Squeeze Theorem that $x_n \to l$ as $n \to \infty$.

Assume now that $x_n \to l$ as $n \to \infty$. We shall show that $y_n \to l$ and $z_n \to l$ as $n \to \infty$. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$|x_n - l| < \frac{\epsilon}{2} \quad \forall n \in \mathbb{N},\ n \ge \mathbb{N},\ \text{i.e.} \tag{44}$$

$$-\frac{\epsilon}{2} < x_n - l < \frac{\epsilon}{2} \quad \forall n \in \mathbb{N},\ n \ge N. \tag{45}$$

It follows from (45) that

$$l - \frac{\epsilon}{2} < x_n < l + \frac{\epsilon}{2} \quad \forall n \in \mathbb{N},\ n \ge N. \tag{46}$$

Using (46) we conclude that $l - \frac{\epsilon}{2}$ is a lower bound for T_N and $l + \frac{\epsilon}{2}$ is an upper bound for T_N. It therefore follows that

$$y_N \ge l - \frac{\epsilon}{2} \tag{47}$$

$$z_N \le l + \frac{\epsilon}{2}. \tag{48}$$

Since $\{y_n\}_{n=1}^{\infty}$ is increasing and $\{z_n\}_{n=1}^{\infty}$ is decreasing we infer from (47), (48) that

$$l - \frac{\epsilon}{2} \le y_N \le y_n \le z_n \le z_N \le l + \frac{\epsilon}{2} \quad \forall n \in \mathbb{N},\ n \ge N. \tag{49}$$

It follows immediately from (49) that

$$|y_n - l| \le \frac{\epsilon}{2} < \epsilon \quad \forall n \in \mathbb{N},\ n \ge N, \tag{50}$$

$$(51) \qquad |z_n - l| \leq \frac{\epsilon}{2} < \epsilon \quad \forall n \in \mathbb{N},\ n \geq N,$$

i.e. $y_n \to l$ and $z_n \to l$ as $n \to \infty$. ■

Proof of II.13: Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence and put $l_s = \limsup_{n\to\infty} x_n$. It follows easily from Lemma II.9 that l_s is a cluster point of $\{x_n\}_{n=1}^{\infty}$. By Proposition II.7, there is a subsequence $\{x_{n_k}\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \to l_s$ as $n \to \infty$. ■

Proof of II.14: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Assume first that $\{x_n\}_{n=1}^{\infty}$ is convergent and put $l = \lim_{n\to\infty} x_n$. Let $\epsilon > 0$ be given and choose $n \in \mathbb{N}$ such that

$$(52) \qquad |x_n - l| < \frac{\epsilon}{2} \quad \forall n \in \mathbb{N},\ n \geq N.$$

Observe that for all $m, n \in \mathbb{N}$ with $m, n \geq N$ we have

$$(53) \qquad x_m - x_n = x_m - l + l - x_n,$$

which yields

$$(54) \qquad |x_m - x_n| \leq |x_m - l| + |l - x_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by virtue of the triangle inequality and (54).

Assume now that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. We shall first show that $\{x_n\}_{n=1}^{\infty}$ is bounded. For this purpose, we choose $N^* \in \mathbb{N}$ such that

$$(55) \qquad |x_m - x_n| < 1 \quad \forall m, n \in \mathbb{N},\ m, n \geq N^*.$$

Put $S = \{|x_1|, |x_2|, \ldots |x_{N^*}|\}$ and let $K = \max\,(S)$. Then, put $M = \max\{K, |x_{N^*}| + 1\}$. Let $n \in \mathbb{N}$ be given. If $n \leq N^*$ then

$$(56) \qquad |x_n| \leq K \leq M.$$

If $n \geq N^*$ then

$$(57) \qquad |x_n| \leq |x_n - x_{N^*}| + |x_{N^*}| < 1 + |x_{N^*}| \leq M.$$

We conclude that $|x_n| \leq M$ for all $n \in \mathbb{N}$, i.e. $\{x_n\}_{n=1}^{\infty}$ is bounded.

By the Bolzano-Weierstrass Theorem we may choose a convergent subsequence $\{x_{n_k}\}_{n=1}^{\infty}$. Let $l = \lim_{k\to\infty} x_{n_k}$. We shall show that $x_n \to l$ as $n \to \infty$. Let $\epsilon > 0$ be given. Choose $K, N \in \mathbb{N}$ such that

(58)
$$|x_{n_k} - l| < \frac{\epsilon}{2} \quad \forall k \in \mathbb{N},\ k \geq K$$

(59)
$$|x_m - x_n| < \frac{\epsilon}{2} \quad \forall m, n \in \mathbb{N},\ m, n \geq N.$$

We choose $k^* \in \mathbb{N}$ such that $k^* \geq K$ and $n_{k^*} \geq N$. (Notice that $k^* = \max\{K, N\}$ will do.) Then, for all $n \in \mathbb{N}$ with $n \geq N$ we have

(60)
$$x_n - l = x_n - x_{n_{k^*}} + x_{n_{k^*}} - l,$$

which gives

(61)
$$|x_n - l| \leq |x_n - x_{n_{k^*}}| + |x_{n_{k^*}} - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by virtue of (58), (59), and the triangle inequality. ■