Department of Mathematical Sciences
Carnegie Mellon University

21-476 **Ordinary Differential Equations** **Fall 2003**

O. Introduction

When modelled mathematically, many real-world phenomena lead to problems in which one would like to determine a function u from an equation involving u and one or more of its derivatives. Such an equation is called a *differential equation*. The unknown function u may be scalar-valued or vector-valued and it may be a function of a single (scalar) variable or of many variables.

In this course we shall study *ordinary differential equations*, i.e. differential equations in which the unknown is a function of a single (real) variable. We shall usually think of the independent variable as being time (and denote it by t). Of course, in many applications, the dependent variable may have a different interpretation. We shall focus our attention primarily on first-order systems of the form

$$\text{(DE)} \qquad \dot{x}(t) = f(t, x(t)),$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function and the unknown x takes values in $\mathbb{R}^n$.

The system (DE) is more general than it may appear at first glance. Indeed, a large class of higher-order equations and systems can be put in the form (DE) by a simple change of variables. For example, the second-order scalar equation

$$\text{(0.1)} \qquad \ddot{\theta}(t) + \alpha\dot{\theta}(t) + \frac{g}{l}\sin\theta(t) = 0,$$

which models the motion of a damped pendulum, can be put in the form (DE) (with $n = 2$) by letting $x_1 = \theta$ and $x_2 = \dot{\theta}$. Similarly the system

$$\text{(0.2)} \qquad \begin{cases} u^{(3)}(t) + \ddot{u}(t) + t\dot{v}(t) - u(t) = e^{3t} \\ \ddot{v}(t) = 2t\dot{v}(t) + t^3u(t) = v(t) = \cos t \end{cases}$$

of two scalar equations can be put in the form (DE) (with $n = 5$) by letting $x_1 = u,\ x_2 = \dot{u},\ x_3 = \ddot{u},\ x_4 = v,\ x_5 = \dot{v}$. Here $u^{(3)}$ is the third derivative of u.]

In applications, solutions of differential equations are typically required to satisfy certain auxiliary conditions, such as initial conditions or boundary conditions. Due to time limitations, the only type of auxiliary conditions that will be discussed in this course are initial conditions, i.e. conditions of the form

$$\text{(IC)} \qquad x(t_0) = x_0$$

where $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ are given (or prescribed) quantities.

In most elementary courses on differential equations the emphasis is placed on recipes for constructing explicit solutions to special types of equations. However, most differential equations arising in applications cannot be solved explicitly in closed form. In more advanced treatments of differential equations the emphasis is usually placed on numerical approximation of solutions or on the use of mathematical tools to study the behavior of solutions without attempting to find explicit formulas for them. This latter approach is referred to as *qualitative theory* of differential equations and will be the central theme of this course. Some typical questions addressed by the qualitative theory are:

1. Given $t_0 \in \mathbb{R},\ x_0 \in \mathbb{R}^n$, is there a solution of (DE) satisfying (IC)?
2. Given $t_0 \in \mathbb{R},\ x_0 \in \mathbb{R}^n$, can there be more than one solution of (DE) satisfying (IC)?
3. Given $t_0 \in \mathbb{R},\ x_0 \in \mathbb{R}^n$, what is the largest interval containing t_0 on which a solution of (DE), (IC) can be defined?
4. If we change $t_0,\ x_0$ by a small amount, how does the solution to (DE), (IC) change?
5. What kind of bounds can we obtain for solutions of (DE)?
6. How do solutions of (DE) behave as $t \to \infty$?
7. Are there any periodic solutions of (DE)?

Of course, numerical analysis and qualitative theory of differential equations cannot and should not be completely separated. In fact, results from the qualitative theory are often used to establish convergence of numerical methods. In addition, many of the algorithms used in numerical computations can also be used to establish properties of solutions. The successful analysis of problems arising in applications generally involves a combination of numerical analysis and qualitative theory.

Although we are not going to emphasize explicit solution techniques in this course, the first section of the notes is devoted to a review of some techniques for solving special classes of first-order scalar equations. The techniques discussed there will be useful in establishing qualitative results for more general classes of equations.