Department of Mathematical Sciences
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21-476 **Ordinary Differential Equations** **Fall 2003**

IV. Existence, Uniqueness, Continuation, and Continuous Dependence

Let D be an open subset of $\mathbb{R} \times \mathbb{R}^n$ and let $f : D \to \mathbb{R}^n$ be given. By a *solution* of the *differential equation*

$$\text{(DE)} \qquad \dot{x}(t) = f(t, x(t)),$$

we mean a differentiable function $x : \mathrm{Dom}(x) \to \mathbb{R}^n$ such that $\mathrm{Dom}(x)$ is an interval with nonempty interior, $Gr(x) \subset D$, and (DE) holds for all $t \in \mathrm{Dom}(x)$. [Here $\mathrm{Dom}(x)$ is the *domain* of x and $Gr(x)$ is the *graph* of x, i.e. $\mathrm{Gr}(x) = \{(t, x(t)) : t \in \mathrm{Dom}(x)\}$.] Given $(t_o, x_o) \in D$, a solution of the *initial-value problem*

$$\text{(IVP)} \qquad \dot{x}(t) = f(t, x(t));\ x(t_o) = x_o,$$

is a solution x of (DE) such that $t_o \in \mathrm{Dom}(x)$ and $x(t_o) = x_o$. We say that f has the *uniqueness property* if for each $(t_o, x_o) \in D$ and every pair x, x^* of solutions of (IVP) we have $x(t) = x^*(t)$ for all $t \in \mathrm{Dom}(x) \cap \mathrm{Dom}(x^*)$. Let x be a solution of (DE). By a *continuation* (or *extension*) of x we mean a solution x^* of (DE) such that $Gr(x) \subset Gr(x^*)$; we say that x^* is a *proper continuation* (or *proper extension*) of x if x^* is a solution of (DE) such that $Gr(x) \subsetneqq Gr(x^*)$. Finally, we say that x is *noncontinuable* (or *inextensible*) if x has no proper continuation.

The following lemma (which is an immediate consequence of the fundamental theorem of calculus) plays an important role in the fundamental theory of (IVP).

Lemma 4.1 *Assume that $f : D \to \mathbb{R}^n$ is continuous and let $(t_o, x_o) \in D$ be given. Let I be an interval with nonempty interior and $t_o \in I$. A continuous function $x : I \to \mathbb{R}^n$ is a solution of (IVP) if and only if x satisfies*

$$\text{(IE)} \qquad x(t) = x_o + \int_{t_o}^{t} f(s, x(s))ds$$

for every $t \in I$.

Theorem 4.2 *(Peano) Assume that f is continuous and let $(t_o, x_o) \in D$ be given. Then there exists $h > 0$ such that (IVP) has at least one solution on $[t_o - h,\ t_o + h]$.*

Continuity of f does not imply the uniqueness property, as the following example shows.

Example 4.3: Let $n = 1$ and consider the initial-value problem

$$\dot{x}(t) = x(t)^{1/3};\ x(0) = 0. \tag{4.1}$$

For each $t_* \geq 0$, define $x_{t^*} : \mathbb{R} \to \mathbb{R}$ by

$$x_{t^*}(t) = \begin{cases} 0, & t \leq t_* \\ \left(\frac{2}{3}(t - t_*)\right)^{3/2}, & t > t_*. \end{cases} \tag{4.2}$$

It is straightforward to show that for each $t_* \geq 0$, x_{t^*} and $-x_{t^*}$ are solutions of (4.1). The zero function is also a solution of (4.1). This initial value problem has uncountably many solutions having the same domain $\mathbb{R}$.

We say that f is *locally Lipschitzean* on D provided that for each closed and bounded set $K \subset D$ there exists $L_K \in \mathbb{R}$ such that

$$||f(t,z) - f(t,y)|| \leq L_K\ ||z - y|| \qquad \text{for all } (t,y),\ (t,z) \in K. \tag{4.3}$$

We say that f is *globally Lipschitzean* on D if there exists $L \in \mathbb{R}$ such that

$$||f(t,z) = f(t,y)|| \leq L\ ||z - y|| \qquad \text{for all } (t,y) \in D. \tag{4.4}$$

Proposition 4.4:

(a) *If the partial derivatives $f_{,2}$, $f_{,3}$, $\ldots f_{,n+1}$ are continuous on D then f is locally Lipschitzean on D.*

(b) *If D is convex and the partial derivatives $f_{,2}$, $f_{,3}$, $\ldots f_{,n+1}$ are bounded and continuous on D then f is globally Lipschitzean on D.*

If f is continuous and locally Lipschitzean on D, then f has the uniqueness property; moreover, the method of *successive approximation* or *Picard iteration* can be used to construct solutions. Given $(t_o, x_o) \in D$ the *Picard iterates* $\{x_{(m)}\}_{m=0}^{\infty}$ for (IVP) are defined recursively as follows:

$$\begin{aligned} x_{(0)}(t) &= x_o \\ x_{(m+1)}(t) &= x_o + \int_{t_o}^{t} f(s, x_{(m)}(s))ds \qquad \text{for all } m \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{4.5}$$

Theorem 4.5 *(Picard-Lindelöf): Assume that f is continuous and locally Lipschitzean on D. Then f has the uniqueness property. Furthermore, for each $(t_o, x_o) \in$*

D there exists $h > 0$ such that the Picard iterates for (IVP) converge uniformly on $[t_o - h,\ t_o + h]$ to the solution of (IVP).

Even if the domain of f is all of $\mathbb{R} \times \mathbb{R}^n$ and x is a noncontinuable solution of (DE) the domain of x may be a proper subset of $\mathbb{R}$, as the following example shows.

Example 4.6: Let $n = 1$. Let $x_o > 0$ be given and consider the initial value problem

$$\dot{x}(t) = x(t)^2;\ x(0) = x_o. \tag{4.6}$$

It is straightforward to check that the function $x : (-\infty,\ 1/x_o) \to \mathbb{R}$ defined by

$$x(t) = \frac{x_o}{1 - tx_o} \qquad \text{for all } t \in (-\infty,\ 1/x_o) \tag{4.7}$$

is a noncontinuable solution of (4.6).

Theorem 4.7: *Assume that f is continuous and let x be a noncontinuable solution of (DE). Them $\mathrm{Dom}(x)$ is an open interval. Furthermore, for each closed and bounded set $K \subset D$ there exist t_*, $t^* \in \mathrm{Dom}(x)$ such that*

$$\begin{aligned} (t, x(t)) &\notin K \qquad \text{for all } t \in \mathrm{Dom}(x) \cap (-\infty,\ t_*) \\ (t, x(t)) &\notin K \qquad \text{for all } t \in \mathrm{Dom}(x) \cap (t^*, \infty). \end{aligned} \tag{4.8}$$

Corollary 4.8: *Let $D = \mathbb{R} \times \mathbb{R}^n$ and assume that f is continuous. Let x be a noncontinuable solution of (DE) with $\mathrm{Dom}(x) = (\eta_-,\ \eta_+)$. If $\eta_- > -\infty$ then $||x(t)|| \to \infty$ as $t \downarrow \eta^-$. If $\eta_+ < \infty$ then $||x(t)|| \to \infty$ as $t \uparrow \eta^+$.*

Theorem 4.9: *Let $D = \mathbb{R} \times \mathbb{R}^n$. Assume that f is continuous and that either f is bounded on $\mathbb{R} \times \mathbb{R}^n$ or f is globally Lipschitzean on $\mathbb{R} \times \mathbb{R}^n$. If x is an inextensible solution of (DE) then $\mathrm{Dom}(x) = (-\infty, \infty)$.*

Theorem 4.10: *Assume that f is continuous and let x be a solution of (DE). Then x has a continuation x^* such that x^* is inextensible.*

Theorem 4.11: *Assume that f is continuous and has the uniqueness property. For each $(t_o,\ x_o) \in D$ let $(\eta_-(t_o,\ x_o),\ \eta_+(t_o,\ x_o))$ denote the domain of the unique noncontinuable solution of (IVP). Define $E \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ by*

$$E = \{(t, t_o, x_o) :\ \eta_-(t_o, x_o) < t < \eta_+(t_o, x_o),\ (t_o, x_o) \in D\}$$

and $\varphi : E \to \mathbb{R}^n$ by $\varphi(t, t_o, x_o)$ is the value at time t of the solution of (IVP). Then E is open and φ is continuous on E.